Physics 364: Problem Set 7
Sean Carroll, Winter 2005
Due Wednesday 2 March, 12:00 noon.

1. (30 points) Recall that “static” means there is a timelike Killing vector which is orthogonal to spacelike hypersurfaces.

(a) Generally speaking, if a vector field $\nu^\mu$ is orthogonal to a set of hypersurfaces defined by $f = \text{constant}$, then we can write the vector as $\nu_\mu = h \nabla_\mu f$ (here both $f$ and $h$ are functions). Show that this implies

$$\nabla_\sigma [\nu_\mu \nabla_\mu \nu_\sigma] = 0.$$  

(b) Imagine we have a perfect fluid with zero pressure (dust), which generates a solution to Einstein’s equations. Show that the metric can be static only if the fluid four-velocity is parallel to the timelike (and hypersurface-orthogonal) Killing vector.

(c) Use Raychaudhuri’s equation to prove that there is no static solution to Einstein’s equations in the presence of a perfect fluid if the pressure is zero and the energy density is greater than zero.

2. (20 points) Consider Einstein’s equations in vacuum, but with a cosmological constant, $G_{\mu\nu} = \Lambda g_{\mu\nu}$.

(a) Solve for the most general static, spherically symmetric metric, in coordinates $(t, r)$ that reduce to ordinary Schwarzschild coordinates when $\Lambda = 0$.

(b) Write down the equation of motion for radial geodesics in terms of an effective potential. Sketch the effective potential for massive particles, and briefly describe its features in words.