Physics 371: Problem Set 6

Sean Carroll, Spring 2006

Due Thursday 18 May, 1:30 p.m.

1. (40 points) Consider a single scalar field ϕ in an anisotropic universe with metric

$$ds^2 = -dt^2 + a^2(t)dx^2 + b^2(t)dy^2 + c^2(t)dz^2 \, .$$

   (a) Derive the equation of motion for ϕ, and the components of the energy-momentum tensor in terms of $\dot{\phi}$ and $\partial_i\phi$ (not assuming homogeneity).

   (b) If ϕ is homogeneous, how does its energy density evolve as the universe expands?

2. (60 points) Consider a theory of two scalar fields, ψ_1 and ψ_2, with a Lagrange density

$$\widehat{\mathcal{L}} = -\frac{1}{2}\sum_i (\nabla_\mu \psi_i)^2 - V(\psi_1, \psi_2) \, ,$$

   where the potential is

$$V(\psi_1, \psi_2) = -\frac{1}{2}m^2(\psi_1^2 + \psi_2^2) + \frac{1}{4}\lambda(\psi_1^2 + \psi_2^2)^2 \, .$$

   Now change to polar coordinates in field space via

$$\phi e^{i\theta} = \psi_1 + i\psi_2 \, .$$

   (a) Rewrite the Lagrange density in terms of the fields ϕ and θ. (For the rest of the problem, work only with these fields, not with ψ_i.)

   (b) Derive the equations of motion for ϕ and θ, both in general and for the specific case of homogeneous fields in a Roberston-Walker metric.

   (c) Derive the energy-momentum tensor of the theory.

   (d) Use the equations of motion to show that the energy-momentum tensor is conserved.

   (e) In Minkowski space, show that there are solutions for which $\phi = \phi_*$ and $\theta = \omega t$, with ϕ_* and ω constant. What is ϕ_* in terms of ω?

   (f) Describe qualitatively what would happen to that solution in an expanding universe, using the equations of motion to justify your answer.