Physics 264: Problem Set 6

Sean Carroll, Fall 2005

Due Thursday 10 November, 12:00 noon

1. (Hartle 8-9; 25 points) Consider the two-dimensional spacetime with metric

$$ds^2 = -X^2 dT^2 + dX^2 \ . \tag{1}$$

   Find the shapes $X(T)$ of all the timelike geodesics in this spacetime.

2. (Hartle 8-11; 25 points) Solve for (all of) the null geodesics in three-dimensional flat spacetime in polar coordinates, with metric

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 \ . \tag{2}$$

   Do light rays move on straight lines?

3. (Hartle 8-14; 25 points) (Fermat's Principle of Least Time.) Consider a medium with an index of refraction $n(x^i)$ that is a function of position. The velocity of light in the medium varies with position and is $c/n(x^i)$. Fermat's principle states that light rays follow paths between two points in *space* (not spacetime) that take the least travel time.

   (a) Show that paths of least time are geodesics in a three-dimensional space with the line element

$$ds^2_{\text{fermat}} = n^2(x^i) ds^2_{\text{flat}} \ , \tag{3}$$

   where $ds^2_{\text{flat}} = dx^2 + dy^2 + dz^2$. (This is not very hard.)

   (b) Write out the three components of the geodesic equation for the extremal paths in (x, y, z) Cartesian coordinates.

4. (Hartle 9-3; 25 points) An observer is stationed at fixed radius R in the Schwarzschild geometry produced by a spherical star of mass M. A proton moving radially outward from the star traverses the observer's laboratory. Its energy E and momentum $|\vec{p}|$ are measured. (Note that the observer doesn't know anything about Schwarzschild coordinates; they measure things according to their own orthonormal basis vectors, with the timelike vector pointing along their staionary worldline.)

   (a) What is the connection between E and $|\vec{p}|$?

   (b) What are the components of the four-momentum of the proton in the Schwarzschild coordinate basis in terms of E and $|\vec{p}|$?