Physics 364: Problem Set 5
Sean Carroll, Winter 2005
Due Wednesday 16 February, 12:00 noon.

1. (15 pts) The metric for the three-sphere in coordinates $x^\mu = (\psi, \theta, \phi)$ can be written

$$ds^2 = d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2) .$$

(a) Calculate the Riemann tensor, Ricci tensor, and curvature scalar.

(b) Show that this metric obeys the relation

$$R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}) ,$$

with the curvature scalar $R$ being a constant, and $n = 3$ in this example. This condition holds if and only if a space is “maximally symmetric” (has the largest possible number of Killing vectors), as discussed in Section 3.9 of the text.

2. (15 pts) Appendix F of the text discusses Raychaudhuri’s equation, which governs the behavior of a congruence (a space-filling set of non-intersecting curves) of geodesics. If a vector field $U^\mu(x)$ describes the set of tangent vectors to such a congruence, Raychaudhuri’s equation says

$$\frac{d\theta}{d\tau} = -\frac{1}{3} \theta^2 - \sigma_{\mu\nu} \sigma^{\mu\nu} + \omega_{\mu\nu} \omega^{\mu\nu} - R_{\mu\nu} U^\mu U^\nu .$$

Here, $\theta$ is the expansion of the congruence, $\sigma_{\mu\nu}$ is the shear of the congruence, and $\omega_{\mu\nu}$ is the rotation of the congruence. In terms of the projection tensor $P_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu$, these quantities are given by

$$\theta = \nabla_\mu U^\mu, \quad \sigma_{\mu\nu} = \nabla_{(\mu} U_{\nu)} - \frac{1}{3} \theta P_{\mu\nu}, \quad \omega_{\mu\nu} = -\nabla_{[\mu} U_{\nu]} .$$

Thus, they are simply the trace, symmetric trace-free, and antisymmetric parts of $\nabla_\mu U_\nu$. Show that, if a set of particles moves along geodesics with zero shear and expansion, then spacetime must have a timelike Killing vector.

3. (20 pts) Consider the following metric, in the coordinates $x^\mu = (u, v, x, y)$:

$$ds^2 = -2dudv + a^2(u)dx^2 + b^2(u)dy^2 ,$$

where $a$ and $b$ are for the moment arbitrary functions of $u$. Physically this metric corresponds to a plane-fronted gravitational wave moving in the $u$ direction. Note that $u$ and $v$ are “null coordinates,” rather than timelike or spacelike; there’s nothing wrong with that.
(a) Calculate the connection coefficients and Riemann tensor for this metric, and calculate the scalar $R_\rho\sigma\mu\nu R^{\rho\sigma\mu\nu}$.

(b) Find a complete set of Killing vector fields on this spacetime. (Hints, which you need not show: there are five Killing vectors in all, and all of them have a vanishing $u$ component $K^u$.)

(c) Introduce functions

$$A(u) = \int^u a^{-2}(u')du', \quad B(u) = \int^u b^{-2}(u')du'.$$

Find the general solution $x^\mu(\lambda)$ to the geodesic equation in this metric, and express your solution in terms of $A$ and $B$. 