Physics 364: Problem Set 5

Sean Carroll, Winter 2005

Due Wednesday 16 February, 12:00 noon.

1. (15 pts) The metric for the three-sphere in coordinates $x^\mu = (\psi, \theta, \phi)$ can be written

$$ds^2 = \mathrm{d}\psi^2 + \sin^2\psi(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2) \ .$$

   (a) Calculate the Riemann tensor, Ricci tensor, and curvature scalar.

   (b) Show that this metric obeys the relation

$$R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}) \ ,$$

   with the curvature scalar R being a constant, and $n = 3$ in this example. This condition holds if and only if a space is "maximally symmetric" (has the largest possible number of Killing vectors), as discussed in Section 3.9 of the text.

2. (15 pts) Appendix F of the text discusses Raychaudhuri's equation, which governs the behavior of a congruence (a space-filling set of non-intersecting curves) of geodesics. If a vector field $U^\mu(x)$ describes the set of tangent vectors to such a congruence, Raychaudhuri's equation says

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}U^\mu U^\nu \ .$$

   Here, θ is the *expansion* of the congruence, $\sigma_{\mu\nu}$ is the *shear* of the congruence, and $\omega_{\mu\nu}$ is the *rotation* of the congruence. In terms of the projection tensor $P_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu$, these quantities are given by

$$\theta = \nabla_\mu U^\mu, \qquad \sigma_{\mu\nu} = \nabla_{(\mu}U_{\nu)} - \frac{1}{3}\theta P_{\mu\nu}, \qquad \omega_{\mu\nu} = -\nabla_{[\mu}U_{\nu]}.$$

   Thus, they are simply the trace, symmetric trace-free, and antisymmetric parts of $\nabla_\mu U_\nu$. Show that, if a set of particles moves along geodesics with zero shear and expansion, then spacetime must have a timelike Killing vector.

3. (20 pts) Consider the following metric, in the coordinates $x^\mu = (u, v, x, y)$:

$$ds^2 = -2\mathrm{d}u\mathrm{d}v + a^2(u)\mathrm{d}x^2 + b^2(u)\mathrm{d}y^2 \ ,$$

   where a and b are for the moment arbitrary functions of u. Physically this metric corresponds to a plane-fronted gravitational wave moving in the u direction. Note that u and v are "null coordinates," rather than timelike or spacelike; there's nothing wrong with that.

(a) Calculate the connection coefficients and Riemann tensor for this metric, and calculate the scalar $R_{\rho\sigma\mu\nu}R^{\rho\sigma\mu\nu}$.

(b) Find a complete set of Killing vector fields on this spacetime. (Hints, which you need not show: there are five Killing vectors in all, and all of them have a vanishing u component K^u.)

(c) Introduce functions

$$A(u) = \int^u a^{-2}(u')du' \ , \qquad B(u) = \int^u b^{-2}(u')du' \ .$$

Find the general solution $x^\mu(\lambda)$ to the geodesic equation in this metric, and express your solution in terms of A and B.