Physics 364: Problem Set 4

Sean Carroll, Winter 2005

Due Wednesday 9 February, 12:00 noon.

1. (15 points) Consider a sphere with coordinates (θ, ϕ) and metric

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2 \ .$$

   (a) Show that lines of constant longitude ($\phi =$ constant) are geodesics, and that the only line of constant latitude ($\theta =$ constant) that is a geodesic is the equator ($\theta = \pi/2$).

   (b) Take a vector at a point p with components $V^\mu(p) = (1, 0)$ and parallel-transport it once around a circle of constant latitude ($\theta =$ constant). What are the components of the resulting vector, as a function of θ?

2. (20 pts) Consider a metric

$$ds^2 = -\left(1 - \frac{R}{r}\right)^2 \mathrm{d}t^2 + \left(1 - \frac{R}{r}\right)^{-2} \mathrm{d}r^2 + r^2\left[\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right] \ . \qquad (0.1)$$

   where r is the radial coordinate and R is some fixed constant.

   (a) Sketch the light cones in the t-r plane (for $r > R$).

   (b) Calculate the Christoffel symbols for this metric.

   (c) Consider a radial electric field,

$$E_r = F_{rt} = f(r) \ . \qquad (0.2)$$

   Find a (nontrivial) function $f(r)$ which solves the curved-spacetime version of Maxwell's equations in this spacetime (with no charges),

$$\nabla_\nu F^{\mu\nu} = 0 \ , \qquad \nabla_{[\sigma} F_{\mu\nu]} = 0 \ . \qquad (0.3)$$

   (d) Find a coordinate transformation $r \to \rho$ such that the metric takes the form

$$ds^2 = -g^{-2}(\rho)\mathrm{d}t^2 + g^2(\rho)\left[\mathrm{d}\rho^2 + \rho^2\left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right)\right] \ , \qquad (0.4)$$

   and solve for $g(\rho)$.

3. (15 points) A good approximation to the metric outside the surface of the Earth is provided by

$$ds^2 = -(1+2\Phi)\mathrm{d}t^2 + (1-2\Phi)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\ ,$$

where the gravitational potential

$$\Phi = -\frac{GM}{r}$$

(familiar from Newtonian gravity) may be assumed to be small. Here G is Newton's constant and M is the mass of the earth.

(a) Imagine there is a clock on the surface of the Earth at distance R_1 from the Earth's center, and another clock on a tall building at distance R_2 from the Earth's center. Calculate the time elapsed on each clock as a function of the coordinate time t. Which clock moves faster?

(b) Solve for a geodesic corresponding to a circular orbit around the equator of the Earth ($\theta = \pi/2$). What is $d\phi/dt$?

(c) How much proper time elapses while a satellite at radius R_1 (skimming along the surface of the earth, neglecting air resistance etc.) completes one orbit? (You can work to first order in Φ if you like.) Plug in the actual numbers for the radius of the Earth and so on to get an answer in seconds. How does this number compare to the proper time elapsed on the clock stationary on the surface?