1. Define $a_{eq}$ as the scale factor and $t_{eq}$ as the time when the density of radiation $\rho_R$ (including massless neutrinos) equals the density of matter $\rho_M$. Assume a flat universe, $k = 0$.

(a) Show that

$$\frac{t}{t_{eq}} = \frac{1}{2 - \sqrt{2}} \left[ \left( \frac{a}{a_{eq}} - 2 \right) \left( \frac{a}{a_{eq}} + 1 \right)^{1/2} + 2 \right].$$

(b) Show that

$$H_{eq} = \frac{4}{3}(\sqrt{2} - 1)t_{eq}^{-1}.$$  

(c) Calculate $1 + z_{eq} = a_0/a_{eq}$ from

$$\rho_{R0} = \frac{\pi^2}{30} g_* T_0^4.$$  

Express your answer in terms of $\Omega_{M0}$ and $h = H_0/(100 \text{ km/sec/Mpc})$. That is, use your knowledge of $T_0$ and $g_*$.  

(d) Calculate the photon temperature $T_{eq}$ in terms of $h$.

2. This problem is considerably simplified in comparison with reality, but it’s enough to get an idea of how things work. We imagine that a massless electron neutrino with 100 GeV of energy scatters off of a proton at rest in our lab, leaving the identity of each particle unaltered. For purposes of this problem, imagine that the proton is a single featureless particle (rather than made up of quarks), and that the weak-interaction vertex comes with a factor $g_w \sim 0.6$ for both the neutrino and the proton.

(a) What is the center-of-mass energy, $E_{cm} = \sqrt{s}$?

(b) What Feynman diagram gives the largest contribution to this process?

(c) Estimate the cross-section using the techniques discussed in class. Express your answer in barns (1 barn = $10^{-24}$ cm$^2$).

(d) Calculate the equivalent cross-section if the particle at rest in our lab had been an electron rather than a proton. Which is larger?
(e) In either case (electron or proton), would the answer have been different if it had been a muon neutrino rather than an electron neutrino?

3. Section 3.4 of Kolb and Turner gives a sketch of how to derive the entropy per comoving volume, \( S = a^3(\rho + p)/T \). Derive the generalization of this formula to the case of a nonzero chemical potential \( \mu \), starting from

\[
TdS = d(\rho V) + pdV - \mu d(nV) .
\]  

(4)

Be explicit.