1. (30 points) Quasar 3C 272 emits relativistic blobs of plasma from near a massive black hole at its center. The blobs travel with speed \( v = \frac{dx}{dt} \) at an angle \( \theta \) with respect to the line-of-sight of the observer. Projected onto the sky, the blobs appear to travel perpendicular to the line of sight with angular speed \( \frac{v_{\text{app}}}{r} \), where \( r \) is the distance to the quasar (treating space as Euclidean, i.e. ignoring the expansion of the universe) and \( v_{\text{app}} \) is the “apparent speed.”

(a) Show that
\[
v_{\text{app}} = \frac{v \sin \theta}{1 - v \cos \theta} \tag{1}
\]

(b) For a given value of \( v \), what value of \( \theta \) maximizes the apparent speed? Can \( v_{\text{app}} \) exceed 1 (the speed of light)?

(c) For 3C 273, \( v_{\text{app}} \approx 10 \). What is the largest value of \( \theta \) in radians?

2. (20 points) A cosmic-ray proton (mass 940 MeV) travels through space at high velocity. If the center-of-mass energy is high enough, it can collide with a cosmic microwave background (CMB) photon (the temperature of the CMB is 2.74K in its overall rest frame) and convert into a proton plus a neutral pion (mass 140 MeV). The pion will then decay into unobservable particles, while the proton will have a lower energy than before the collision. What is the cosmic-ray energy above which we expect this process to occur, and therefore provide a cutoff in the cosmic-ray spectrum? (This is known as the Griesen-Zatsepin-Kuzmin, or GZK, cutoff. In fact there are observational indications that it is violated, which might be a sign of new physics — even, some have suggested, a violation of special relativity.)

3. (50 points) A common occurence in particle physics is the scattering of two particles \( A + B \) into two new particles \( C + D \). For such events it is convenient to define Mandelstam variables:
\[
s = -\eta_{\mu\nu}(p_A^\mu + p_B^\mu)(p_A^\nu + p_B^\nu) \\
t = -\eta_{\mu\nu}(p_A^\mu - p_C^\mu)(p_A^\nu - p_C^\nu) \\
u = -\eta_{\mu\nu}(p_A^\mu - p_D^\mu)(p_A^\nu - p_D^\nu)
\]
where $p_i^\mu$ are the 4-momenta. The beauty of these variables is that they are all Lorentz-invariant.

(a) Show that $s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$.

(b) Express the energy of $A$ in the center-of-mass (CM) frame (in which the spatial components of the total momentum vanish), in terms of the masses and the Mandelstam variables.

(c) Express the energy of $A$ in the “lab” frame, in which $B$ is at rest.

(d) Express the total energy in the CM frame.

(e) For scattering of identical particles, $A + A \rightarrow A + A$, show that in the CM frame we have

\[
\begin{align*}
    s &= 4(p^0 + m_A^2) \\
    t &= -2p^2(1 - \cos \theta) \\
    u &= -2p^2(1 + \cos \theta),
\end{align*}
\]

where $p$ is the 3-momentum of one of the incident particles, and $\theta$ is the scattering angle.