Physics 364: Problem Set 2

Sean Carroll, Winter 2005

Due Wednesday 26 January, 12:00 noon.

1. For a system of discrete point particles in Minkowski space the energy-momentum tensor takes the form
$$T_{\mu\nu} = \sum_a \frac{p_\mu^{(a)} p_\nu^{(a)}}{p^{0(a)}} \delta^{(3)}(\mathbf{x} - \mathbf{x}^{(a)}) \ ,$$
where the index a labels the different particles. Show that, for a dense collection of particles with isotropically distributed velocities, we can smooth over the individual particle worldlines to obtain the perfect-fluid energy-momentum tensor
$$T^{\mu\nu} = (\rho + p) U^\mu U^\nu + p\eta^{\mu\nu} \ .$$

2. Show that the two-dimensional torus T^2 is a manifold, by explicitly constructing an appropriate atlas.

3. The commutator of two vector fields is defined by
$$[X, Y](f) \equiv X(Y(f)) - Y(X(f)) \ .$$
   (a) Verify that the commutator of any two vector fields is itself a vector field, by showing that its action on functions is linear and obeys the Leibniz rule.

   (b) Show explicitly that the component formula
$$[X, Y]^\mu = X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu$$
   tranforms like a vector under general coordinate transformations.

   (c) Since partials commute, the commutator of the vector fields given by the partial derivatives of coordinate functions, $\{\partial_\mu\}$, always vanishes. Give an example of two linearly independent, nowhere-vanishing vector fields in R^2 whose commutator does not vanish. Notice that these fields provide a basis for the tangent space at each point, but it cannot be a coordinate basis since the commutator doesn't vanish.

4. Prolate spheroidal coordinates can be used to simplify the Kepler problem in celestial mechanics. They are related to the usual Cartesian coordinates (x, y, z) of Euclidean

3-space by

$$
\begin{aligned}
x &= \sinh\chi\,\sin\theta\,\cos\phi\,, \\
y &= \sinh\chi\,\sin\theta\,\sin\phi\,, \\
z &= \cosh\chi\,\cos\theta\,.
\end{aligned}
$$

Restrict your attention to the plane $y = 0$ and answer the following questions.

(a) What is the coordinate transformation matrix $\partial x^{\mu}/\partial x^{\nu'}$ relating (x, z) to (χ, θ)?

(b) What does the line element ds^2 look like in prolate spheroidal coordinates?