Physics 371: Problem Set 1

Sean Carroll, Spring 2006

Due Thursday 13 April, 1:30 p.m.

1. (40 points) The point here is to get you comfortable with the pushing around of indices. Consider a universe that looks like the flat Robertson-Walker metric, but is *anisotropic*, in the sense of having different scale factors for each of the three spatial directions:

$$\begin{aligned} ds^2 &= -dt^2 + a_1^2(t)dx^2 + a_2^2(t)dy^2 + a_3^2(t)dz^2 \\ &= -dt^2 + \sum_i a_i^2(t)(dx^i)^2 \,. \end{aligned}$$

   (a) Calculate the Christoffel symbols $\Gamma^\rho_{\mu\nu}$.

   (b) Calculate the Riemann tensor $R^\rho{}_{\sigma\mu\nu}$, Ricci tensor $R_{\mu\nu}$, and curvature scalar R.

   (c) Imagine that the pressure in the x^i direction is denoted p_i, so that the energy-momentum tensor has components

$$T_{\mu\nu} = \begin{pmatrix} \rho & & & \\ & a_1^2 p_1 & & \\ & & a_2^2 p_2 & \\ & & & a_3^2 p_3 \end{pmatrix} . \tag{1}$$

   What equations do you derive from the Einstein equation? (*I.e.*, what are the equivalents of the Friedmann equations?)

2. (30 points) In class we briefly mentioned the existence of "horizons" in cosmology – the past light cone of an event can intersect the big bang at a finite distance, so there will be particles whose worldlines do not intersect that light cone.

   (a) Consider two particles at the epoch of recombination, when the microwave background was formed, at a redshift $z = 1200$. Imagine that the two particles are just outside each other's horizon; that is, the past light cones just touch at the big bang. Imagine further that the universe has been flat and matter-dominated for its whole history (not true, but imagine it). What is the presently observed angular separation of these two points on the sky?

   (b) Now imagine a universe which is flat and has been matter dominated ever since some redshift z_*, but before that it was vacuum-dominated. That is, imagine that that all of the energy density in the universe was vacuum ($p = -\rho$) up to z_*, then suddenly turned into matter ($p = 0$) at a phase transition (nothing wrong with that). Show that the past light cones of any two points will intersect in the past.

3. (30 points) In cosmology we tend to idealize non-relativistic particles as having zero temperature T and pressure p. In reality, random motions will give them some temperature and pressure, satisfying $p \propto T\rho$.

    (a) How does the pressure of a gas of massive particles decay as a function of the scale factor? (Note the discussion in section 2.2 of Kolb and Turner, where they show that the physical three-velocity of a nonrelativistic particle evolves as $1/a$.)

    (b) Suppose neutrinos have a mass $m_\nu = 1$ eV, and a current temperature almost that of the CMB photons, $T_{\nu 0} = 2\text{K}$. At about what redshift did the neutrinos go from being relativistic to non-relativistic?