Physics 364: Problem Set 1

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Due Wednesday 19 January, 12:00 noon.

1. Imagine that space (not spacetime) is actually a finite box, or in more sophisticated lingo, a three-torus, of size L. By this we mean that there is a coordinate system $x^\mu = (t, x, y, z)$ such that every point with coordinates (t, x, y, z) is *identified* with every point with coordinates $(t, x + L, y, z)$, $(t, x, y + L, z)$, or $(t, x, y, z + L)$. Note that the time coordinate is the same. Now consider two observers; observer A is at rest in this coordinate system (constant spatial coordinates), while observer B moves in the x-direction with constant velocity v. A and B begin at the same event, and while A remains still B moves once around the universe and comes back to intersect the worldline of A without ever having to accelerate (since the universe is periodic). What are the relative proper times experienced in this interval by A and B? Is this consistent with your understanding of Lorentz invariance?

2. A cosmic-ray proton (mass 940 MeV) travels through space at high velocity. If the center-of-mass energy is high enough, it can collide with a cosmic microwave background (CMB) photon (the temperature of the CMB is 2.74K in its overall rest frame) and convert into a proton plus a neutral pion (mass 140 MeV). The pion will then decay into unobservable particles, while the proton will have a lower energy than before the collision. What is the cosmic-ray energy above which we expect this process to occur, and therefore provide a cutoff in the cosmic-ray spectrum? (This is known as the Griesen-Zatsepin-Kuzmin, or GZK, cutoff. In fact there are observational indications that it is violated, which might be a sign of new physics — even, some have suggested, a violation of special relativity.)

3. A common occurence in particle physics is the scattering of two particles $A+B$ into two new particles $C + D$. For such events it is convenient to define *Mandelstam variables*:

$$\begin{aligned} s &= -\eta_{\mu\nu}(p^\mu_A + p^\mu_B)(p^\nu_A + p^\nu_B) \\ t &= -\eta_{\mu\nu}(p^\mu_A - p^\mu_C)(p^\nu_A - p^\nu_C) \\ u &= -\eta_{\mu\nu}(p^\mu_A - p^\mu_D)(p^\nu_A - p^\nu_D)\ , \end{aligned}$$

where p^μ_i are the 4-momenta. The beauty of these variables is that they are all Lorentz-invariant.

(a) Show that $s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$.

(b) Express the energy of A in the center-of-mass (CM) frame (in which the spatial components of the total momentum vanish), in terms of the masses and the Mandelstam variables.

(c) Express the energy of A in the "lab" frame, in which B is at rest.

(d) Express the total energy in the CM frame.

(e) For scattering of identical particles, $A + A \to A + A$, show that in the CM frame we have

$$\begin{aligned} s &= 4(\mathbf{p}^2 + m_A^2) \\ t &= -2\mathbf{p}^2(1 - \cos\theta) \\ u &= -2\mathbf{p}^2(1 + \cos\theta) \ , \end{aligned}$$

where $\mathbf{p}$ is the 3-momentum of one of the incident particles, and θ is the scattering angle.

4. Using the tensor transformation law applied to $F_{\mu\nu}$, show how the electric and magentic field 3-vectors $\mathbf{E}$ and $\mathbf{B}$ transform under

(a) a rotation about the y-axis,

(b) a boost along the z-axis.

5. Consider Maxwell's electromagnetism with $J^\mu = 0$. The equations of motion are

$$\partial_\nu F^{\mu\nu} = 0 \ , \qquad \partial_{[\mu} F_{\nu\sigma]} = 0$$

and the energy-momentum tensor is

$$T^{\mu\nu} = F^{\mu\lambda} F^{\nu}{}_{\lambda} - \frac{1}{4}\eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \ . \tag{0.1}$$

(a) Express the components of the energy-momentum tensor in three-vector notation, using the divergence, curl, electric and magnetic fields, and an overdot to denote time derivatives.

(b) Using the equations of motion, verify that the energy-momentum tensor is conserved.