8 Cosmology

Contemporary cosmological models are based on the idea that the universe is pretty much the same everywhere — a stance sometimes known as the **Copernican principle**. On the face of it, such a claim seems preposterous; the center of the sun, for example, bears little resemblance to the desolate cold of interstellar space. But we take the Copernican principle to only apply on the very largest scales, where local variations in density are averaged over. Its validity on such scales is manifested in a number of different observations, such as number counts of galaxies and observations of diffuse X-ray and γ-ray backgrounds, but is most clear in the 3° microwave background radiation. Although we now know that the microwave background is not perfectly smooth (and nobody ever expected that it was), the deviations from regularity are on the order of 10^{-5} or less, certainly an adequate basis for an approximate description of spacetime on large scales.

The Copernican principle is related to two more mathematically precise properties that a manifold might have: isotropy and homogeneity. **Isotropy** applies at some specific point in the space, and states that the space looks the same no matter what direction you look in. More formally, a manifold M is isotropic around a point p if, for any two vectors V and W in T_pM, there is an isometry of M such that the pushforward of W under the isometry is parallel with V (not pushed forward). It is isotropy which is indicated by the observations of the microwave background.

Homogeneity is the statement that the metric is the same throughout the space. In other words, given any two points p and q in M, there is an isometry which takes p into q. Note that there is no necessary relationship between homogeneity and isotropy; a manifold can be homogeneous but nowhere isotropic (such as $\mathbf{R} \times S^2$ in the usual metric), or it can be isotropic around a point without being homogeneous (such as a cone, which is isotropic around its vertex but certainly not homogeneous). On the other hand, if a space is isotropic *everywhere* then it is homogeneous. (Likewise if it is isotropic around one point and also homogeneous, it will be isotropic around every point.) Since there is ample observational evidence for isotropy, and the Copernican principle would have us believe that we are not the center of the universe and therefore observers elsewhere should also observe isotropy, we will henceforth assume both homogeneity and isotropy.

There is one catch. When we look at distant galaxies, they appear to be receding from us; the universe is apparently not static, but changing with time. Therefore we begin construction of cosmological models with the idea that the universe is homogeneous and isotropic in space, but not in time. In general relativity this translates into the statement that the universe can be foliated into spacelike slices such that each slice is homogeneous and isotropic.

We therefore consider our spacetime to be $\mathbf{R} \times \Sigma$, where $\mathbf{R}$ represents the time direction and Σ is a homogeneous and isotropic three-manifold. The usefulness of homogeneity and isotropy is that they imply that Σ must be a maximally symmetric space. (Think of isotropy as invariance under rotations, and homogeneity as invariance under translations. Then homogeneity and isotropy together imply that a space has its maximum possible number of Killing vectors.) Therefore we can take our metric to be of the form

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}(u)\mathrm{d}u^i\mathrm{d}u^j \ . \tag{8.1}$$

Here t is the timelike coordinate, and (u^1, u^2, u^3) are the coordinates on Σ; γ_{ij} is the maximally symmetric metric on Σ. This formula is a special case of (7.2), which we used to derive the Schwarzschild metric, except we have scaled t such that $g_{tt} = -1$. The function $a(t)$ is known as the **scale factor**, and it tells us "how big" the spacelike slice Σ is at the moment t. The coordinates used here, in which the metric is free of cross terms $\mathrm{d}t\,\mathrm{d}u^i$ and the spacelike components are proportional to a single function of t, are known as **comoving coordinates**, and an observer who stays at constant u^i is also called "comoving". Only a comoving observer will think that the universe looks isotropic; in fact on Earth we are not quite comoving, and as a result we see a dipole anisotropy in the cosmic microwave background as a result of the conventional Doppler effect.

Our interest is therefore in maximally symmetric Euclidean three-metrics γ_{ij}. We know that maximally symmetric metrics obey

$${}^{(3)}R_{ijkl} = k(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}) \ , \tag{8.2}$$

where k is some constant, and we put a superscript ${}^{(3)}$ on the Riemann tensor to remind us that it is associated with the three-metric γ_{ij}, not the metric of the entire spacetime. The Ricci tensor is then

$${}^{(3)}R_{jl} = 2k\gamma_{jl} \ . \tag{8.3}$$

If the space is to be maximally symmetric, then it will certainly be spherically symmetric. We already know something about spherically symmetric spaces from our exploration of the Schwarzschild solution; the metric can be put in the form

$$d\sigma^2 = \gamma_{ij}\mathrm{d}u^i\,\mathrm{d}u^j = e^{2\beta(r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \ . \tag{8.4}$$

The components of the Ricci tensor for such a metric can be obtained from (7.16), the Ricci tensor for a spherically symmetric spacetime, by setting $\alpha = 0$ and $\partial_0\beta = 0$, which gives

$$\begin{aligned}
{}^{(3)}R_{11} &= \frac{2}{r}\partial_1\beta \\
{}^{(3)}R_{22} &= e^{-2\beta}(r\partial_1\beta - 1) + 1 \\
{}^{(3)}R_{33} &= [e^{-2\beta}(r\partial_1\beta - 1) + 1]\sin^2\theta \ .
\end{aligned} \tag{8.5}$$

We set these proportional to the metric using (8.3), and can solve for $\beta(r)$:

$$\beta = -\frac{1}{2}\ln(1 - kr^2) \ . \tag{8.6}$$

This gives us the following metric on spacetime:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right] \ . \tag{8.7}$$

This is the **Robertson-Walker metric**. We have not yet made use of Einstein's equations; those will determine the behavior of the scale factor $a(t)$.

Note that the substitutions

$$\begin{aligned} k &\to \tfrac{k}{|k|} \\ r &\to \sqrt{|k|}\,r \\ a &\to \tfrac{a}{\sqrt{|k|}} \end{aligned} \tag{8.8}$$

leave (8.7) invariant. Therefore the only relevant parameter is $k/|k|$, and there are three cases of interest: $k = -1$, $k = 0$, and $k = +1$. The $k = -1$ case corresponds to constant negative curvature on Σ, and is called **open**; the $k = 0$ case corresponds to no curvature on Σ, and is called **flat**; the $k = +1$ case corresponds to positive curvature on Σ, and is called **closed**.

Let us examine each of these possibilities. For the flat case $k = 0$ the metric on Σ is

$$\begin{aligned} d\sigma^2 &= \mathrm{d}r^2 + r^2 d\Omega^2 \\ &= \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \ , \end{aligned} \tag{8.9}$$

which is simply flat Euclidean space. Globally, it could describe $\mathbf{R}^3$ or a more complicated manifold, such as the three-torus $S^1 \times S^1 \times S^1$. For the closed case $k = +1$ we can define $r = \sin\chi$ to write the metric on Σ as

$$d\sigma^2 = \mathrm{d}\chi^2 + \sin^2\chi\, d\Omega^2 \ , \tag{8.10}$$

which is the metric of a three-sphere. In this case the only possible global structure is actually the three-sphere (except for the non-orientable manifold $\mathbf{R}\mathrm{P}^3$). Finally in the open $k = -1$ case we can set $r = \sinh\psi$ to obtain

$$d\sigma^2 = \mathrm{d}\psi^2 + \sinh^2\psi\, d\Omega^2 \ . \tag{8.11}$$

This is the metric for a three-dimensional space of constant negative curvature; it is hard to visualize, but think of the saddle example we spoke of in Section Three. Globally such a space could extend forever (which is the origin of the word "open"), but it could also describe a non-simply-connected compact space (so "open" is really not the most accurate description).

With the metric in hand, we can set about computing the connection coefficients and curvature tensor. Setting $\dot{a} \equiv da/dt$, the Christoffel symbols are given by

$$\begin{aligned}
\Gamma^0_{11} &= \frac{a\dot{a}}{1-kr^2} \qquad \Gamma^0_{22} = a\dot{a}r^2 \qquad \Gamma^0_{33} = a\dot{a}r^2\sin^2\theta \\
\Gamma^1_{01} &= \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = \frac{\dot{a}}{a} \\
\Gamma^1_{22} &= -r(1-kr^2) \qquad \Gamma^1_{33} = -r(1-kr^2)\sin^2\theta \\
\Gamma^2_{12} &= \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} \\
\Gamma^2_{33} &= -\sin\theta\cos\theta \qquad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta \ .
\end{aligned} \tag{8.12}$$

The nonzero components of the Ricci tensor are

$$\begin{aligned}
R_{00} &= -3\frac{\ddot{a}}{a} \\
R_{11} &= \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1-kr^2} \\
R_{22} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k) \\
R_{33} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k)\sin^2\theta \ ,
\end{aligned} \tag{8.13}$$

and the Ricci scalar is then

$$R = \frac{6}{a^2}(a\ddot{a} + \dot{a}^2 + k) \ . \tag{8.14}$$

The universe is not empty, so we are not interested in vacuum solutions to Einstein's equations. We will choose to model the matter and energy in the universe by a perfect fluid. We discussed perfect fluids in Section One, where they were defined as fluids which are isotropic in their rest frame. The energy-momentum tensor for a perfect fluid can be written

$$T_{\mu\nu} = (p+\rho)U_\mu U_\nu + pg_{\mu\nu} \ , \tag{8.15}$$

where ρ and p are the energy density and pressure (respectively) as measured in the rest frame, and U^μ is the four-velocity of the fluid. It is clear that, if a fluid which is isotropic in some frame leads to a metric which is isotropic in some frame, the two frames will coincide; that is, the fluid will be at rest in comoving coordinates. The four-velocity is then

$$U^\mu = (1,0,0,0) \ , \tag{8.16}$$

and the energy-momentum tensor is

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{ij}p & \\ 0 & & & \end{pmatrix} \ . \tag{8.17}$$

With one index raised this takes the more convenient form

$$T^{\mu}{}_{\nu} = \mathrm{diag}(-\rho, p, p, p) \ . \tag{8.18}$$

Note that the trace is given by

$$T = T^{\mu}{}_{\mu} = -\rho + 3p \ . \tag{8.19}$$

Before plugging in to Einstein's equations, it is educational to consider the zero component of the conservation of energy equation:

$$\begin{aligned} 0 &= \nabla_\mu T^{\mu}{}_{0} \\ &= \partial_\mu T^{\mu}{}_{0} + \Gamma^{\mu}_{\mu 0} T^{0}{}_{0} - \Gamma^{\lambda}_{\mu 0} T^{\mu}{}_{\lambda} \\ &= -\partial_0 \rho - 3\frac{\dot{a}}{a}(\rho + p) \ . \end{aligned} \tag{8.20}$$

To make progress it is necessary to choose an **equation of state**, a relationship between ρ and p. Essentially all of the perfect fluids relevant to cosmology obey the simple equation of state

$$p = w\rho \ , \tag{8.21}$$

where w is a constant independent of time. The conservation of energy equation becomes

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \ , \tag{8.22}$$

which can be integrated to obtain

$$\rho \propto a^{-3(1+w)} \ . \tag{8.23}$$

The two most popular examples of cosmological fluids are known as **dust** and **radiation**. Dust is collisionless, nonrelativistic matter, which obeys $w = 0$. Examples include ordinary stars and galaxies, for which the pressure is negligible in comparison with the energy density. Dust is also known as "matter", and universes whose energy density is mostly due to dust are known as **matter-dominated**. The energy density in matter falls off as

$$\rho \propto a^{-3} \ . \tag{8.24}$$

This is simply interpreted as the decrease in the number density of particles as the universe expands. (For dust the energy density is dominated by the rest energy, which is proportional to the number density.) "Radiation" may be used to describe either actual electromagnetic radiation, or massive particles moving at relative velocities sufficiently close to the speed of light that they become indistinguishable from photons (at least as far as their equation of state is concerned). Although radiation is a perfect fluid and thus has an energy-momentum tensor given by (8.15), we also know that $T_{\mu\nu}$ can be expressed in terms of the field strength as

$$T^{\mu\nu} = \frac{1}{4\pi}(F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}g^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}) \ . \tag{8.25}$$

The trace of this is given by

$$T^{\mu}{}_{\mu} = \frac{1}{4\pi}\left[F^{\mu\lambda}F_{\mu\lambda} - \frac{1}{4}(4)F^{\lambda\sigma}F_{\lambda\sigma}\right] = 0 \ . \tag{8.26}$$

But this must also equal (8.19), so the equation of state is

$$p = \frac{1}{3}\rho \ . \tag{8.27}$$

A universe in which most of the energy density is in the form of radiation is known as **radiation-dominated**. The energy density in radiation falls off as

$$\rho \propto a^{-4} \ . \tag{8.28}$$

Thus, the energy density in radiation falls off slightly faster than that in matter; this is because the number density of photons decreases in the same way as the number density of nonrelativistic particles, but individual photons also lose energy as a^{-1} as they redshift, as we will see later. (Likewise, massive but relativistic particles will lose energy as they "slow down" in comoving coordinates.) We believe that today the energy density of the universe is dominated by matter, with $\rho_{\rm mat}/\rho_{\rm rad} \sim 10^6$. However, in the past the universe was much smaller, and the energy density in radiation would have dominated at very early times.

There is one other form of energy-momentum that is sometimes considered, namely that of the vacuum itself. Introducing energy into the vacuum is equivalent to introducing a cosmological constant. Einstein's equations with a cosmological constant are

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu} \ , \tag{8.29}$$

which is clearly the same form as the equations with no cosmological constant but an energy-momentum tensor for the vacuum,

$$T^{(\rm vac)}_{\mu\nu} = -\frac{\Lambda}{8\pi G}g_{\mu\nu} \ . \tag{8.30}$$

This has the form of a perfect fluid with

$$\rho = -p = \frac{\Lambda}{8\pi G} \ . \tag{8.31}$$

We therefore have $w = -1$, and the energy density is independent of a, which is what we would expect for the energy density of the vacuum. Since the energy density in matter and radiation decreases as the universe expands, if there is a nonzero vacuum energy it tends to win out over the long term (as long as the universe doesn't start contracting). If this happens, we say that the universe becomes **vacuum-dominated**.

We now turn to Einstein's equations. Recall that they can be written in the form (4.45):

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) . \tag{8.32}$$

The $\mu\nu = 00$ equation is

$$-3\frac{\ddot{a}}{a} = 4\pi G(\rho + 3p) , \tag{8.33}$$

and the $\mu\nu = ij$ equations give

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi G(\rho - p) . \tag{8.34}$$

(There is only one distinct equation from $\mu\nu = ij$, due to isotropy.) We can use (8.33) to eliminate second derivatives in (8.34), and do a little cleaning up to obtain

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) , \tag{8.35}$$

and

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} . \tag{8.36}$$

Together these are known as the **Friedmann equations**, and metrics of the form (8.7) which obey these equations define Friedmann-Robertson-Walker (FRW) universes.

There is a bunch of terminology which is associated with the cosmological parameters, and we will just introduce the basics here. The rate of expansion is characterized by the **Hubble parameter**,

$$H = \frac{\dot{a}}{a} . \tag{8.37}$$

The value of the Hubble parameter at the present epoch is the Hubble constant, H_0. There is currently a great deal of controversy about what its actual value is, with measurements falling in the range of 40 to 90 km/sec/Mpc. ("Mpc" stands for "megaparsec", which is 3×10^{24} cm.) Note that we have to divide $\dot{a}$ by a to get a measurable quantity, since the overall scale of a is irrelevant. There is also the **deceleration parameter**,

$$q = -\frac{a\ddot{a}}{\dot{a}^2} , \tag{8.38}$$

which measures the rate of change of the rate of expansion.

Another useful quantity is the **density parameter**,

$$\Omega = \frac{8\pi G}{3H^2}\rho$$

$$= \frac{\rho}{\rho_{\rm crit}} , \tag{8.39}$$

where the **critical density** is defined by

$$\rho_{\rm crit} = \frac{3H^2}{8\pi G} . \tag{8.40}$$

This quantity (which will generally change with time) is called the "critical" density because the Friedmann equation (8.36) can be written

$$\Omega - 1 = \frac{k}{H^2 a^2} . \tag{8.41}$$

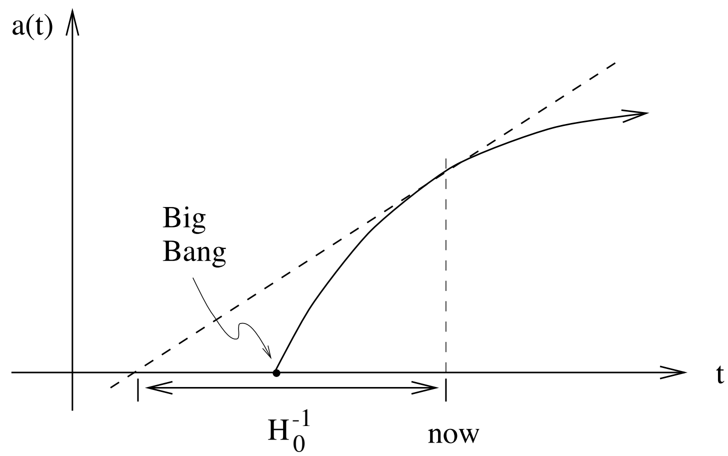
The sign of k is therefore determined by whether Ω is greater than, equal to, or less than one. We have

$$\begin{array}{ccccccc} \rho < \rho_{\rm crit} & \leftrightarrow & \Omega < 1 & \leftrightarrow & k = -1 & \leftrightarrow & \text{open} \\ \rho = \rho_{\rm crit} & \leftrightarrow & \Omega = 1 & \leftrightarrow & k = 0 & \leftrightarrow & \text{flat} \\ \rho > \rho_{\rm crit} & \leftrightarrow & \Omega > 1 & \leftrightarrow & k = +1 & \leftrightarrow & \text{closed} . \end{array}$$

The density parameter, then, tells us which of the three Robertson-Walker geometries describes our universe. Determining it observationally is an area of intense investigation.

It is possible to solve the Friedmann equations exactly in various simple cases, but it is often more useful to know the qualitative behavior of various possibilities. Let us for the moment set $\Lambda = 0$, and consider the behavior of universes filled with fluids of positive energy ($\rho > 0$) and nonnegative pressure ($p \geq 0$). Then by (8.35) we must have $\ddot{a} < 0$. Since we know from observations of distant galaxies that the universe is expanding ($\dot{a} > 0$), this means that the universe is "decelerating." This is what we should expect, since the gravitational attraction of the matter in the universe works against the expansion. The fact that the universe can only decelerate means that it must have been expanding even faster in the past; if we trace the evolution backwards in time, we necessarily reach a singularity at $a = 0$. Notice that if $\ddot{a}$ were exactly zero, $a(t)$ would be a straight line, and the age of the universe would be H_0^{-1}. Since $\ddot{a}$ is actually negative, the universe must be somewhat younger than that.

This singularity at $a = 0$ is the **Big Bang**. It represents the creation of the universe from a singular state, not explosion of matter into a pre-existing spacetime. It might be hoped that the perfect symmetry of our FRW universes was responsible for this singularity, but in fact it's not true; the singularity theorems predict that any universe with $\rho > 0$ and $p \geq 0$ must have begun at a singularity. Of course the energy density becomes arbitrarily high as $a \to 0$, and we don't expect classical general relativity to be an accurate description of nature in this regime; hopefully a consistent theory of quantum gravity will be able to fix things up.



The future evolution is different for different values of k. For the open and flat cases, $k \leq 0$, (8.36) implies

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 + |k| \ . \tag{8.42}$$

The right hand side is *strictly* positive (since we are assuming $\rho > 0$), so $\dot{a}$ never passes through zero. Since we know that today $\dot{a} > 0$, it must be positive for all time. Thus, the open and flat universes expand forever — they are temporally as well as spatially open. (Please keep in mind what assumptions go into this — namely, that there is a nonzero positive energy density. Negative energy density universes do not have to expand forever, even if they are "open".)

How fast do these universes keep expanding? Consider the quantity ρa^3 (which is constant in matter-dominated universes). By the conservation of energy equation (8.20) we have

$$\begin{aligned} \frac{d}{dt}(\rho a^3) &= a^3\left(\dot{\rho} + 3\rho\frac{\dot{a}}{a}\right) \\ &= -3pa^2\dot{a} \ . \end{aligned} \tag{8.43}$$

The right hand side is either zero or negative; therefore

$$\frac{d}{dt}(\rho a^3) \leq 0 \ . \tag{8.44}$$

This implies in turn that ρa^2 must go to zero in an ever-expanding universe, where $a \to \infty$. Thus (8.42) tells us that

$$\dot{a}^2 \to |k| \ . \tag{8.45}$$

(Remember that this is true for $k \leq 0$.) Thus, for $k = -1$ the expansion approaches the limiting value $\dot{a} \to 1$, while for $k = 0$ the universe keeps expanding, but more and more slowly.

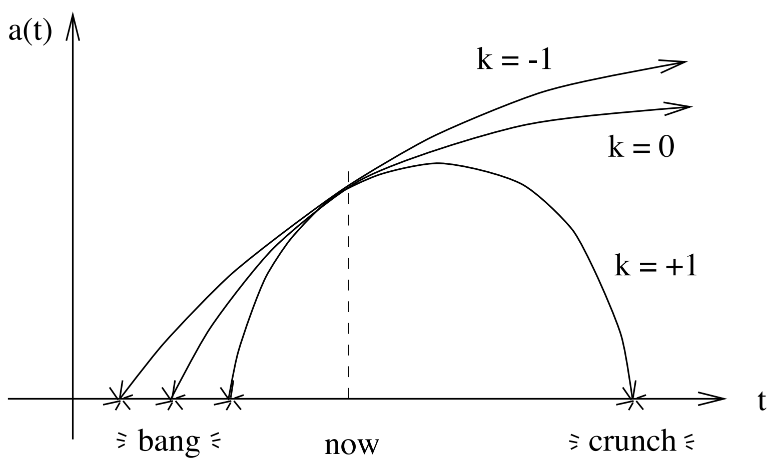
For the closed universes ($k = +1$), (8.36) becomes

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - 1 \ . \tag{8.46}$$

The argument that $\rho a^2 \to 0$ as $a \to \infty$ still applies; but in that case (8.46) would become negative, which can't happen. Therefore the universe does not expand indefinitely; a possesses an upper bound $a_{\rm max}$. As a approaches $a_{\rm max}$, (8.35) implies

$$\ddot{a} \to -\frac{4\pi G}{3}(\rho + 3p)a_{\rm max} < 0 \ . \tag{8.47}$$

Thus $\ddot{a}$ is finite and negative at this point, so a reaches $a_{\rm max}$ and starts decreasing, whereupon (since $\ddot{a} < 0$) it will inevitably continue to contract to zero — the Big Crunch. Thus, the closed universes (again, under our assumptions of positive ρ and nonnegative p) are closed in time as well as space.



We will now list some of the exact solutions corresponding to only one type of energy density. For dust-only universes ($p = 0$), it is convenient to define a **development angle** $\phi(t)$, rather than using t as a parameter directly. The solutions are then, for open universes,

$$\begin{cases} a = \frac{C}{2}(\cosh\phi - 1) \\ t = \frac{C}{2}(\sinh\phi - \phi) \end{cases} \qquad (k = -1) \ , \tag{8.48}$$

for flat universes,

$$a = \left(\frac{9C}{4}\right)^{1/3} t^{2/3} \qquad (k = 0) \ , \tag{8.49}$$

and for closed universes,

$$\begin{cases} a = \frac{C}{2}(1 - \cos\phi) \\ t = \frac{C}{2}(\phi - \sin\phi) \end{cases} \qquad (k = +1) \ , \tag{8.50}$$

where we have defined

$$C = \frac{8\pi G}{3}\rho a^3 = \text{ constant } . \tag{8.51}$$

For universes filled with nothing but radiation, $p = \frac{1}{3}\rho$, we have once again open universes,

$$a = \sqrt{C'}\left[\left(1 + \frac{t}{\sqrt{C'}}\right)^2 - 1\right]^{1/2} \qquad (k = -1) \ , \tag{8.52}$$

flat universes,

$$a = (4C')^{1/4} t^{1/2} \qquad (k = 0) \ , \tag{8.53}$$

and closed universes,

$$a = \sqrt{C'}\left[1 - \left(1 - \frac{t}{\sqrt{C'}}\right)^2\right]^{1/2} \qquad (k = +1) \ , \tag{8.54}$$

where this time we defined

$$C' = \frac{8\pi G}{3}\rho a^4 = \text{ constant } . \tag{8.55}$$

You can check for yourselves that these exact solutions have the properties we argued would hold in general.

For universes which are empty save for the cosmological constant, either ρ or p will be negative, in violation of the assumptions we used earlier to derive the general behavior of $a(t)$. In this case the connection between open/closed and expands forever/recollapses is lost. We begin by considering $\Lambda < 0$. In this case Ω is negative, and from (8.41) this can only happen if $k = -1$. The solution in this case is

$$a = \sqrt{\frac{-3}{\Lambda}}\sin\left(\sqrt{\frac{-\Lambda}{3}}\,t\right) \ . \tag{8.56}$$

There is also an open $(k = -1)$ solution for $\Lambda > 0$, given by

$$a = \sqrt{\frac{3}{\Lambda}}\sinh\left(\sqrt{\frac{\Lambda}{3}}\,t\right) \ . \tag{8.57}$$

A flat vacuum-dominated universe must have $\Lambda > 0$, and the solution is

$$a \propto \exp\left(\pm\sqrt{\frac{\Lambda}{3}}\,t\right) \ , \tag{8.58}$$

while the closed universe must also have $\Lambda > 0$, and satisfies

$$a = \sqrt{\frac{3}{\Lambda}}\cosh\left(\sqrt{\frac{\Lambda}{3}}\,t\right) \ . \tag{8.59}$$

These solutions are a little misleading. In fact the three solutions for $\Lambda > 0$ — (8.57), (8.58), and (8.59) — all represent the same spacetime, just in different coordinates. This spacetime, known as **de Sitter space**, is actually maximally symmetric as a spacetime. (See Hawking and Ellis for details.) The $\Lambda < 0$ solution (8.56) is also maximally symmetric, and is known as **anti-de Sitter space**.

It is clear that we would like to observationally determine a number of quantities to decide which of the FRW models corresponds to our universe. Obviously we would like to determine H_0, since that is related to the age of the universe. (For a purely matter-dominated, $k = 0$ universe, (8.49) implies that the age is $2/(3H_0)$. Other possibilities would predict similar relations.) We would also like to know Ω, which determines k through (8.41). Given the definition (8.39) of Ω, this means we want to know both H_0 and ρ_0. Unfortunately both quantities are hard to measure accurately, especially ρ. But notice that the deceleration parameter q can be related to Ω using (8.35):

$$\begin{aligned} q &= -\frac{a\ddot{a}}{\dot{a}^2} \\ &= -H^{-2}\frac{\ddot{a}}{a} \\ &= \frac{4\pi G}{3H^2}(\rho + 3p) \\ &= \frac{4\pi G}{3H^2}\rho(1 + 3w) \\ &= \frac{1 + 3w}{2}\Omega\ . \end{aligned} \tag{8.60}$$

Therefore, if we think we know what w is (*i.e.*, what kind of stuff the universe is made of), we can determine Ω by measuring q. (Unfortunately we are not completely confident that we know w, and q is itself hard to measure. But people are trying.)

To understand how these quantities might conceivably be measured, let's consider geodesic motion in an FRW universe. There are a number of spacelike Killing vectors, but no timelike Killing vector to give us a notion of conserved energy. There is, however, a Killing tensor. If $U^\mu = (1, 0, 0, 0)$ is the four-velocity of comoving observers, then the tensor

$$K_{\mu\nu} = a^2(g_{\mu\nu} + U_\mu U_\nu) \tag{8.61}$$

satisfies $\nabla_{(\sigma}K_{\mu\nu)} = 0$ (as you can check), and is therefore a Killing tensor. This means that if a particle has four-velocity $V^\mu = dx^\mu/d\lambda$, the quantity

$$K^2 = K_{\mu\nu}V^\mu V^\nu = a^2[V_\mu V^\mu + (U_\mu V^\mu)^2] \tag{8.62}$$

will be a constant along geodesics. Let's think about this, first for massive particles. Then we will have $V_\mu V^\mu = -1$, or

$$(V^0)^2 = 1 + |\vec{V}|^2\ , \tag{8.63}$$

where $|\vec{V}|^2 = g_{ij}V^iV^j$. So (8.61) implies

$$|\vec{V}| = \frac{K}{a} \ . \tag{8.64}$$

The particle therefore "slows down" with respect to the comoving coordinates as the universe expands. In fact this is an actual slowing down, in the sense that a gas of particles with initially high relative velocities will cool down as the universe expands.

A similar thing happens to null geodesics. In this case $V_\mu V^\mu = 0$, and (8.62) implies

$$U_\mu V^\mu = \frac{K}{a} \ . \tag{8.65}$$

But the frequency of the photon as measured by a comoving observer is $\omega = -U_\mu V^\mu$. The frequency of the photon emitted with frequency ω_1 will therefore be observed with a lower frequency ω_0 as the universe expands:

$$\frac{\omega_0}{\omega_1} = \frac{a_1}{a_0} \ . \tag{8.66}$$

Cosmologists like to speak of this in terms of the **redshift** z between the two events, defined by the fractional change in wavelength:

$$\begin{aligned} z &= \frac{\lambda_0 - \lambda_1}{\lambda_1} \\ &= \frac{a_0}{a_1} - 1 \ . \end{aligned} \tag{8.67}$$

Notice that this redshift is not the same as the conventional Doppler effect; it is the expansion of space, not the relative velocities of the observer and emitter, which leads to the redshift.

The redshift is something we can measure; we know the rest-frame wavelengths of various spectral lines in the radiation from distant galaxies, so we can tell how much their wavelengths have changed along the path from time t_1 when they were emitted to time t_0 when they were observed. We therefore know the ratio of the scale factors at these two times. But we don't know the times themselves; the photons are not clever enough to tell us how much coordinate time has elapsed on their journey. We have to work harder to extract this information.

Roughly speaking, since a photon moves at the speed of light its travel time should simply be its distance. But what is the "distance" of a far away galaxy in an expanding universe? The comoving distance is not especially useful, since it is not measurable, and furthermore because the galaxies need not be comoving in general. Instead we can define the **luminosity distance** as

$$d_L^2 = \frac{L}{4\pi F} \ , \tag{8.68}$$

where L is the absolute luminosity of the source and F is the flux measured by the observer (the energy per unit time per unit area of some detector). The definition comes from the fact that in flat space, for a source at distance d the flux over the luminosity is just one over the area of a sphere centered around the source, $F/L = 1/A(d) = 1/4\pi d^2$. In an FRW universe, however, the flux will be diluted. Conservation of photons tells us that the total number of photons emitted by the source will eventually pass through a sphere at comoving distance r from the emitter. Such a sphere is at a physical distance $d = a_0 r$, where a_0 is the scale factor when the photons are observed. But the flux is diluted by two additional effects: the individual photons redshift by a factor $(1+z)$, and the photons hit the sphere less frequently, since two photons emitted a time δt apart will be measured at a time $(1+z)\delta t$ apart. Therefore we will have

$$\frac{F}{L} = \frac{1}{4\pi a_0^2 r^2 (1+z)^2} , \tag{8.69}$$

or

$$d_L = a_0 r(1+z) . \tag{8.70}$$

The luminosity distance d_L is something we might hope to measure, since there are some astrophysical sources whose absolute luminosities are known ("standard candles"). But r is not observable, so we have to remove that from our equation. On a null geodesic (chosen to be radial for convenience) we have

$$0 = ds^2 = -\mathrm{d}t^2 + \frac{a^2}{1-kr^2}\mathrm{d}r^2 , \tag{8.71}$$

or

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_0^r \frac{dr}{(1-kr^2)^{1/2}} . \tag{8.72}$$

For galaxies not too far away, we can expand the scale factor in a Taylor series about its present value:

$$a(t_1) = a_0 + (\dot{a})_0(t_1 - t_0) + \frac{1}{2}(\ddot{a})_0(t_1 - t_0)^2 + \ldots . \tag{8.73}$$

We can then expand both sides of (8.72) to find

$$r = a_0^{-1}\left[(t_0 - t_1) + \frac{1}{2}H_0(t_0 - t_1)^2 + \ldots\right] . \tag{8.74}$$

Now remembering (8.67), the expansion (8.73) is the same as

$$\frac{1}{1+z} = 1 + H_0(t_1 - t_0) - \frac{1}{2}q_0 H_0^2(t_1 - t_0)^2 + \ldots . \tag{8.75}$$

For small $H_0(t_1 - t_0)$ this can be inverted to yield

$$t_0 - t_1 = H_0^{-1}\left[z - \left(1 + \frac{q_0}{2}\right)z^2 + \ldots\right] . \tag{8.76}$$

Substituting this back again into (8.74) gives

$$r = \frac{1}{a_0 H_0}\left[z - \frac{1}{2}\left(1 + q_0\right) z^2 + \ldots\right] . \tag{8.77}$$

Finally, using this in (8.70) yields **Hubble's Law**:

$$d_L = H_0^{-1}\left[z + \frac{1}{2}(1 - q_0) z^2 + \ldots\right] . \tag{8.78}$$

Therefore, measurement of the luminosity distances and redshifts of a sufficient number of galaxies allows us to determine H_0 and q_0, and therefore takes us a long way to deciding what kind of FRW universe we live in. The observations themselves are extremely difficult, and the values of these parameters in the real world are still hotly contested. Over the next decade or so a variety of new strategies and more precise application of old strategies could very well answer these questions once and for all.