Can primodial inflation explain why the universe is accelerating today?

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Abstract

Super-horizon gravitaional perturbation was proposed to be a possible explanation for the accelerating expansion of the universe. However, without invoking the dark energy, the theory was still proven to be non-valid.

1 Introduction

Under years of observation, people come to the well-known assumption that the universe is homogeneous and isotropic, which can be discribed by the Friedmann-Robertson-Walker (FRW) cosmological model. However, recent observations have also shown that the universe apears to be under accelerating expansion which can not be explained by matter-dominated FRW theory. For general relativity to be considered describing our universe correctly, one must invoke new kind of mechanism to account for this effect. Cosmological constant is one possible explanation and the other possibility is the dark energy which has negative pressure in the energy momentum tensor. The later one suggests new physics which still have no direct evidence yet. Recently, there were proposals [1] pointing out that if superhorizon perturbations exist and vary with time, one might account their effect as the accelerating expansion history of the universe. The author of [1] considered this to be able to explain the acceleration without adding unkown physical objects. However, it was later pointed out that under the frame of GR and strong energy condition, this is not possible. This report gives a dicussion of the idea in [1] and why it doesn't work.

2 The Claim

We start from the synchronous and comoving gauge, where the world is filled with non-relativistic, pressureless, irrotational dust, or say CDM particles. In this synchronous-comoving frame, the 4-vector of the CDM particle is $u^\mu = (1, 0, 0, 0)$ and covariant 4-vector is $u_\mu = (-1, 0, 0, 0)$. Thus, from $u^\mu = g^{\mu\nu} u_\nu$, the line metric element must have the form

$$ds^2 = -dt^2 + a^2 \gamma_{ij} dx^i dx^j, \tag{1}$$

where γ_{ij} is the spatial matric element to be calculated from the perturbation of the Poisson equation. The calculation of the metric purterbations is long and we will just quote the results here [2]. To first order, with the Poisson equation

$$\nabla^2 \varphi = \frac{k^2}{2} a^2 \rho^{(0)} \delta^{(1)},$$

we have

$$\gamma_{ij} = (1 - 2\psi^{(1)})\delta_{ij} + D_{ij}\chi^{(1)}, \tag{2}$$

where $D_{ij} = \partial_i \partial_j - \frac{1}{3}\nabla^2 \delta_{ij}$, and the vector and tensor modes are ignored for they do not arise from the inflation mechanism. And

$$\begin{aligned} \psi^{(1)}(x,t) &= \frac{5}{3}\varphi(x) + \frac{t^2}{18}\nabla^2\varphi(x) && (3)\\ &= \frac{5}{3}\varphi(x) + \frac{2}{81}a(t)H_0^{-2}\nabla^2\varphi(x) && (4)\\ \chi^{(1)}(x,t) &= -\frac{4}{27}a(t)H_0^{-2}\varphi(x). && (5) \end{aligned}$$

Where we have used the fact that $a \propto t^{2/3}$ and $H_0 = 2/3t^{-1}$. And the second order term is

$$\begin{aligned} \psi^{(2)}(x,t) = &- \frac{50}{9}\varphi^2 - \frac{5t^2}{54}\varphi^{,k}\varphi_{,k} \\ &+ \frac{t^4}{252}[(\nabla^2\varphi)^2 - \frac{10}{3}\varphi^{,ki}\varphi_{,ki}] \end{aligned} \tag{6}$$

In the original paper, the spatial matric was written in the Riemann normal coordinates, where to leading order

$$x^i = e^{-10\varphi^2(0)/3} y^i. \tag{7}$$

thus

$$\begin{aligned}\gamma_{ij}(y) &= e^{-20\varphi^2(0)/3}\gamma_{ij}(x)\\ &= e^{-20\varphi^2(0)/3}(1-2\psi^{(1)}+\frac{2}{3}\nabla^2\chi^{(1)})\delta_{ij}\\ &\approx exp\left\{-\frac{10}{3}\varphi(y)-\frac{4}{9}e^{10\varphi(y)/3}\nabla^2\varphi(y)/H_0^2\right\}.\end{aligned} \tag{8}$$

Up to a factor which could be rescaled into $a(t)$. Thus, the line element could be written in the form

$$ds^2 = -dt^2 + a^2(t)e^{-2\Psi(x,t)}\delta_{ij}dx^i dx^j, \tag{9}$$

rewrite x as the Riemann normal coordinate and Ψ is the perturbation we just obtained above.

$$\Psi(x,t) = \frac{5}{3}\varphi(x) + \frac{2}{9}e^{10\varphi(x)/3}\nabla^2\varphi(x)/H_0^2 \tag{10}$$

In order to see how the large scale perturbation affects the expansion rate, we split the potential Ψ into two parts: $\Psi = \Psi_l + \Psi_s$, where Ψ_l contains perturbations from the super-Hubble modes and Ψ_s has the contribution from the modes of wavelengths smaller than the Hubble radius. As we have mentioned, the contribution from the super-Hubble modes will not be detected by the local observer and, instead, will be counted as part of the new scale factor

$$\bar{a}(t) = a(t)e^{-\Psi_l(t)+\Psi_l(0)}. \tag{11}$$

Where $\Psi_l(0)$ is the value of present time. and we could take $\bar{a_0} = a_0 = 1$. And the resulting new line element becomes

$$ds^2 = -dt^2 + \bar{a}^2(t)e^{-2\Psi_s(x,t)}\delta_{ij}dx^i dx^j, \tag{12}$$

Except for the small scale inhomogeneities brought in by the Ψ_s modes, the universe still looks like an homogeneous FRW universe when the large scale modes are treated as background. The new Hubble constant under this perturbation is now

$$\bar{H} = \frac{\dot{\bar{a}}}{\bar{a}} = H - \dot{\Psi}_l \tag{13}$$

thus the Super-Hubble modes enter the expansion rate. By performing the same "operational" definition for the deceleration parameter q,

$$q = -\frac{\ddot{\bar{a}}}{\bar{a}\bar{H}^2}, \tag{14}$$

will lead to the modified value

$$\begin{aligned}\bar{q} &= -1 - \frac{\dot{\bar{H}}}{\bar{H}^2} \\ &= -1 + \frac{3/2 + \ddot{\Psi}_l/H^2}{(1-\dot{\Psi}_l/H)^2} \end{aligned} \tag{15}$$

From eq. (10), after taking out the sub-Hubble scale perturbation and absorb the time , we can write

$$\Psi_l = a(t)\frac{2}{9}\frac{e^{10\varphi(x)/3}\nabla^2\varphi(x)}{H_0^2} \equiv a(t)\Psi_{l0} \tag{16}$$

Put this into eq.(15) and eq.(13), and by using the fact that $a \propto t^{2/3}$, we obtain

$$\begin{aligned}\bar{H} &= \frac{\dot{a}}{a} - \dot{a}\Psi \\ &= H_0(a^{-3/2} - a^{-1/2}\Psi_{l0}) \\ &= \frac{\bar{H}_0}{1-\Psi_{l0}}(a^{-3/2} - a^{-1/2}\Psi_{l0}),\end{aligned}$$

and

$$\begin{aligned}\bar{q} &= -1 + \frac{3/2 + \ddot{a}/H^2\Psi_{l0}}{(1-\dot{a}/H\Psi_{l0})^2} \\ &= -1 + \frac{3/2 - aq\Psi_{l0}}{(1-a\Psi_{l0})^2} \\ &= -1 + \frac{3/2 - a\Psi_{l0}/2}{(1-a\Psi_{l0})^2}.\end{aligned} \tag{18}$$

When no perturbation is present, $\bar{q} = 1/2$, the universe is decelerating. With the norelativistic matter perturbation, the potential Ψ_{l0} is actually affecting the deceleration constant. As it was claimed by the author, the value of Ψ_{l0} in our Hubble volume can acctually taking any of the possible statistical values. To estimate the value of Ψ_{l0}, one calculate the spatial variance $\langle\Psi_{l0}\rangle \approx \sqrt{Var[exp(10\varphi/3)\nabla^2\varphi]}/H_0^2$. And this value should depend largely on the statistical nature of the perturbation. Even if φ itself is small, $\nabla^2\varphi$ might be large in our seemingly small patch of Hubble volume and can bring the total value to the oder of unity. If, so, the deceleration constant might be less than zero.

3 Why it is wrong

Now the question is whether this claim is valid or not? One disproof is from the basis of general relativity.

3.1 From strong energy condition

In order to examine the validity of the claim, we follow the calculation of [3]. In general relativity, the meausre of expansion over local geodesic lines can be discribed by the *expansion parameter* θ, with the definition $\theta = \theta^{\mu}_{\mu}$ and $\theta_{\mu\nu} \equiv \nabla_{\mu} u_{\nu}$. Here, u^{μ} is, like we did in the previous section, chosen to be the 4-velocity of the CDM particles of interests and thus θ measures the average expansion of the local CDM geodesics.

θ can be decomposed into three irreducible parts

$$\theta_{\mu\nu} = \frac{1}{3}(g_{\mu\nu} + u_{\mu}u_{\nu}) + \omega_{\mu\nu} + \sigma_{\mu\nu}, \tag{19}$$

where $\omega_{\mu\nu} = \theta_{[\mu,\nu]}$ is the antisymmetric tensor which measures the rotation while $\sigma_{\mu\nu} = \theta_{\{\mu,\nu\}} - \frac{1}{3}\theta(g_{\mu\nu} + u_{\mu}u_{\nu})$ is the symmetric tensor measuring the shear.

How θ is related to q? Define the local Hubble constant

$$\bar{H} = \frac{1}{3}\nabla_{\mu} u^{\mu}. \tag{20}$$

To see if this definition makes sense, consider in the synchronous and comoving frame, $u^{\mu} = (1, 0, 0, 0)$, thus

$$\bar{H} = \frac{1}{3}(\partial_{\mu} + \Gamma^{\mu}_{\mu\nu})u^{\nu} = \frac{1}{3}\Gamma^{\mu}_{\mu 0} \tag{21}$$

A simple comparison can be carried out, since in the no-perturbation case, $\Gamma^{\mu}_{\mu 0} = 3\dot{a}/a$. Thus $\bar{H}$ reduces to the nonperturbed form $H = \dot{a}/a$. .

We can also define,

$$q = -\frac{\bar{a}\ddot{\bar{a}}}{\dot{\bar{a}}^2} = -1 - \frac{\dot{\bar{H}}}{\bar{H}^2} = -1 - \bar{H}^{-2} u^{\mu}\nabla_{\mu}\bar{H}. \tag{22}$$

Here, we have proper time derivative $d/d\tau = u^{\mu}\nabla_{\mu}$. In [3], the definition above was also checked to be equivalent to eq.(18).

In eq. (22),

$$u^{\mu}\nabla_{\mu}\bar{H} = \frac{1}{3}\frac{d\theta}{d\tau} \tag{23}$$

And by Raychaudhuri's equation,

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}u^{\mu}u^{\nu}, \tag{24}$$

where $R_{\mu\nu}$ is the Ricci tensor. Thus, from eq. (22) to (24),

$$q = \frac{1}{3}(\sigma_{\mu\nu}\sigma^{\mu\nu} - \omega_{\mu\nu}\omega^{\mu\nu} + R_{\mu\nu}u^\mu u^\nu)/\bar{H}^2 \tag{25}$$

By using Einstein's equation,

$$\begin{aligned} R_{\mu\nu}u^\mu u^\nu &= (8\pi G T_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R)u^\mu u^\nu \\ &= 8\pi G(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)u^\mu u^\nu \\ &= 8\pi G(T_{\mu\nu}u^\mu u^\nu + \frac{1}{2}T) \end{aligned} \tag{26}$$

If the *strong energy condition* should hold, which is the case when we assume no dark energy should present, then this term should be positive semidefinite. On the other hand, by the definition of θ, we have $\theta_{\mu\nu}u^\mu = \theta_{\mu\nu}u^\nu = 0$. This means that in the comoving frame, $u^\mu = (t, 0, 0, 0)$, $\theta_{\mu\nu}$ only has spatial part. Thus, terms like $\sigma_{\mu\nu}\sigma^{\mu\nu}$ and $\omega_{\mu\nu}\omega^{\mu\nu}$ will also be nonnegitive, since we can alwas perform teh Lorentz transformation to the comoving frame to do the contraction.

Now, whether q will be negitive or not depends on the second term $\omega_{\mu\nu}\omega^{\mu\nu}$. Since we worked on the synchronous CDM-comoving frame $u^\mu = (1, 0, 0, 0)$, which implies that

$$\begin{aligned} \omega_{\mu\nu} &= \theta_{[\mu,\nu]} \\ &= \nabla_\mu u_\nu - \nabla_\nu u_\mu = 0. \end{aligned} \tag{27}$$

Thus, $q \geq 0$ holds. That is, without violating the strong energy condition or introducing vorticity ($\omega \neq 0$), the presence of superhorizon perturbation from cosmological poisson equation still couldn't explain the acceleration of the universe expansion.

3.2 From spatial curvature renormalization

More specifically, we could also try to see whether $\varphi\nabla^2\varphi$ will lead to the deceleration or not. One way is by comparing the perturbation with the renormalization of the local spatial curvature [5].

Consider the FRW metric with curvature k such that

$$ds^2 = -dt^2 + a^2(t)(1 + \frac{1}{4}kr^2)^{-2}\delta ij dx^i dx^j, \tag{28}$$

Here, k could be taken as any value. However, if we rescale the coordinate, we could renormalize it back to the common value -1, 0 and +1. By solving the Friedmann equation, one easily gets

$$q = \frac{1}{2}(1 + \frac{k}{a^2 H^2}) \tag{29}$$

Now turn back to what we just got in the first and second order perturbation of the γ_{ij} matice, and work on the comoving frame. By dropping terms of higher order than $\nabla^2\varphi$ in the matrice element and expand φ aournd the observation point, we have

$$\varphi \approx \varphi_0 + \frac{1}{6}\nabla^2\varphi r^2 \tag{30}$$

where φ_0 is a sptial constant wihch corresponds to the superhorrizon mode. Here, we also assumed that φ is isotropic around this point. Put this into γ_{ij}, we have

$$\gamma_{ij} = \left\{1 - \frac{10}{3}\varphi_0 + \frac{50}{9}\varphi_0^2 - \frac{t^2}{9}\nabla^2\varphi - \frac{5}{9}\nabla^2\varphi r^2 + \frac{50}{27}\varphi_0\nabla^2\varphi r^2\right\}\delta_{ij}. \tag{31}$$

Again, renormalize the scale factor to the form

$$\bar{a} = a[1 - \frac{10}{3}\varphi_0 + \frac{50}{9}\varphi_0^2 - \frac{t^2}{9}\nabla^2\varphi]^{1/2} \tag{32}$$

And do the expansion for the γ_{ij}, it is shown that the $\varphi_0\nabla^2\varphi$ terms actually cancel out. By ignoring higher order terms

$$\gamma_{ij} = (1 - \frac{5}{9}\nabla^2\varphi r^2)\delta_{ij} \tag{33}$$

And by comparing with eq. (28), the equivalent curvature is

$$k = \frac{10}{9}\nabla^2\varphi \tag{34}$$

for the superhorizon modes.

With this curvature term and put it back to eq.(29), we could calculate the modified expansion rate resulting from the local curvature variation. Since

$$q = \frac{1}{2}(1 + \frac{10\nabla^2\varphi}{9\dot{\bar{a}}^2}) \tag{35}$$

and from eq.(32), after the expansion and ignoring terms of orders higher than $\varphi_0\nabla^2\varphi$ and $(\nabla^2\varphi)^2$, thus we obtain

$$q = \frac{1}{2}[1 + (\frac{5}{18}\nabla^2\varphi + \frac{25}{27}\varphi_0\nabla^2\varphi)(\frac{2}{\dot{a}})^2] \tag{36}$$

The second term $\varphi_0 \nabla^2 \varphi$ is what was claimed to be having possible large fluctuations in the prvious section. However, from this calculation, it is more likely to originate from the correction of the curvature renormalization. On the other hand, from Friedmann equation,

$$2H^2 q = H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \geq 0 \tag{37}$$

When classical matters are present, the deceleration parameter simply can not be negative! Thus, theres is no chance for the acceleration to happen for only classical objcts in present.

4 Conclusion

In this paper, I discussed how Kold et. al. think super-horizon perturbation could lead to acceleration expansion and also showed two ways to disprove it. The first method is much stronger than the second one since it is a more general proof based on the validity of GR and strong energy condition. Unless eighter GR, SEC or the homogeneity of the universe is broken, the acceleration of the expansion might not possibly exist.

References

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