Constraints on deviation of G_N during nucleosynthesis from recent analysis

Shane Caldwell

2 June 2006

1 Introduction

The work of Bambi *et al* [1] is aimed at understanding how different values of Newton's constant would affect the primordial abundances of light elements, and thereby While the title of the paper alludes to a general modification of G_N, it must be noted that there are important constraints on the rate of change of G_N that restrict us to consideration of only slow variation compared to the concurrent rate of expansion.

2 Overview of Primordial Nucleosynthesis

Nucleosynthesis is the process of nuclear reactions which produce the lightest nuclear species (^{2}H,^{3}H, ^{3}He and ^{4}He, and ^{7}Li) out of an existing population of protons and neutrons. The theory of big bang nucleosynthesis (BBN) is especially critical to cosmology because it incorporates diverse parts of the standard cosmology (for example, the expansion rate H, the ratio $\eta \equiv n_N/n_\gamma$ of nucleons to photons, and the neutron lifetime τ_n) in making reasonably definite, testable predictions about the abundances of the light elements we should expect to observe now.

In predicting the abundances emerging from the process of BBN we focus on two critical stages that take place during different epochs in the radiation-dominated era of the Standard Cosmology. The first stage corresponds to the temperature regime between 0.2 MeV and 2 MeV and determines the relative populations of protons and neutrons out of which BBN proceeds. It therefore sets the initial conditions for nucleosynthesis. The second stage is the actual process of nucleosynthesis and corresponds to temperatures between 0.02 MeV and 0.2 MeV, low enough so that the high energy tail of the blackbody distribution of photons sufficiently decouples from light nuclei to allow them to survive after formation.

The first stage begins with free nucleons in thermal and chemical equilibrium in the plasma. Their equilibrium is maintained by the interconversion of neutrons with protons via the weak interactions

$$\begin{aligned} \nu_e + n &\leftrightarrow p + e^- \\ e^+ + n &\leftrightarrow p + \bar{\nu}_e \\ n &\leftrightarrow p + e^- + \bar{\nu}_e. \end{aligned}$$

We are assuming that all nucleonic matter is free at this stage. Technically there are equilibrium abundances of light nuclei for this temperature regime, but they are smaller than the abundance of free nucleons by tens of orders of magnitude [4]. It should be noted that our program of separating the freeze-out stage from the onset of nucleosynthesis is sensible only because the entropy per nucleon ($s/n_N \sim 10^1 1$) is so high as to ensure that while free nucleons remain in equilibrium the abundances of the light elements are negligible.

Following the semi-analytic treatment of Bernstein *et al* [2] we consider these weak interactions in terms of aggregates, letting $\lambda_{pn}(t)$ represent the aggregate rate of reactions converting protons to neutrons, and $\lambda_{np}(t)$ the aggregate rate of the reverse reactions. These rates depend on time t implicitly their dependence on temperature T while

$$t = \frac{0.301}{\sqrt{g_\star}} M_{Pl} T^{-2} \tag{1}$$

where $g_\star$ is the effective number of relativistic degrees of freedom. We can write a differential equation for the fractional abundance $X_n = n_n/(n_n + n_p) \equiv n_n/n_N$ of neutrons (subscript n) relative to nucleons (subscript N), simply by imposing balance on the conversion reactions:

$$\frac{dX_n}{dt} = \lambda_{pn}(t) \left[1 - X_n(t)\right] - \lambda_{np}(t) X_n(t). \tag{2}$$

By setting the left hand side of this to zero we see immediately that in chemical equilibrium the neutron fraction is

$$X_n^{eq}(t) = \frac{\lambda_{pn}(t)}{\lambda_{pn}(t) + \lambda_{np}(t)} \equiv \frac{\lambda_{pn}(t)}{\Gamma_{n\leftrightarrow p}} = \left[1 + e^{\Delta_{np}/T(t)}\right]^{-1} \tag{3}$$

where the last equality comes from the detailed balance $\lambda_{pn} = \lambda_{np} e^{-\Delta_{np}/T}$, with $\Delta_{np} \equiv m_n - m_p = 1.29$ MeV the mass difference between neutron and proton. The general solution to the balance equation (2) can be approximated by [6]

$$X_n \simeq X_n^{eq} \left[1 + \frac{H}{\Gamma_{n\leftrightarrow p}} \frac{d \ln X_n^{eq}}{d \ln T}\right]. \tag{4}$$

This equation means that the neutron abundance follows its equilibrium value until the interconversion rate $\Gamma_{n\leftrightarrow p}$ has fallen to the scale of the expansion rate H. When this happens, the weak interactions become frozen out and the reactants fall out of chemical equilibrium. A rough approximation for the temperature T_F at which this happens comes by equating $H \sim \Gamma_{n\leftrightarrow p} \sim G_F^2 T^5$ and using $H = 1.66\sqrt{g_\star}T^2/M_{Pl}$ to obtain

$$T_f \sim \left(\frac{\sqrt{g_\star}}{G_F^2 M_{Pl}}\right)^{1/3} \sim g_\star^{1/6} \times 1 \text{ MeV} \tag{5}$$

corresponding to a freeze-out time of $t_f \approx 1$ sec. (Note that at this stage we have $g_\star \approx 10.75$.) Roughly approximating the neutron abundance at t_f by its equilibrium value then gives

$$X_n(T_f) \simeq X_n^{eq}(T_f) = \left[1 + e^{\Delta_{np}/T_f}\right]^{-1}. \tag{6}$$

Since the exponent $\Delta_{np}/T_f \sim 1$ we conclude that a significant minority of nucleons were neutrons when the nucleonic species decoupled from one another at T_f. (This is a remarkable conspiracy of nature, since the exponent is sensitive to the strong and electromagnetic forces via Δ_{np} and to the gravitational and weak forces via T_f.) To properly calculate the asymptotically surviving neutron abundance $X_n(t \to \infty)$ from freeze-out requires the explicitly computing the individual rate of each of the three neutron decay reactions above [7], which finally yields [6]

$$X_n(t \to \infty) = 0.150 \tag{7}$$

which actually obtains by $T \approx 0.25$ MeV or $t \approx 20$ s.

The freezing out of the weak interactions *mostly* fixes the neutron abundance at this value until the second critical stage of nucleosynthesis, which is when the fractional abundance X_D of deuterium becomes appreciable. Deuterium is created by the reaction $n + p \to D + \gamma$ which has a rate, per nucleon, of $4.55 \times 10^{-20} n_p$ cm-1 s^{-1} [7], which exceeds the expansion rate down to temperatures on the order of 1 KeV, well out of the scope of our project. The $p(n,\gamma)$D reaction is overcome, however, by its reverse reaction (the photodissociation of deuterium by an energetic photon) until several minutes after the freeze-out of the weak interactions. To find this time t_d we examine the equilibrium distribution of deuterium determined by the Saha equation, which accounts for the competition of reactions,

$$\frac{n_D}{n_n n_p} = \frac{g_D}{g_n g_p}\left(\frac{m_D}{m_n m_p}\right)^{3/2}\left(\frac{T}{2\pi}\right)^{-3/2} e^{\Delta_D/T} \tag{8}$$

where $\Delta_D \equiv m_n + m_p - M_D = 2.23$ MeV is the binding energy of deuterium, g_i are the statistical spin weights $g_p = g_n = 2, g_D = 3$. This can be rewritten in terms of the mass fractions $X_i \equiv n_i A_i / n_N$ as [6]

$$\frac{X_D}{X_n X_p} \simeq \frac{24\zeta(3)}{\sqrt{\pi}} \left(\frac{T_d}{m_p}\right) \eta \, e^{\Delta_D/T}. \tag{9}$$

We approximate T_d by demanding that the normalized fraction $\frac{X_D}{X_n X_p}$ of deuterium be of order 1. Then the last equation has the solution $T_d \sim 0.1$ MeV. By a more careful treatment, developing approximate solutions to the rate equations involving deuterium, Bernstein *et al* give $T_d \approx 0.86$ MeV, corresponding to $t_d \approx 180$ s, since now $g_\star \approx 3.36$.

Once there is an appreciable amount of deuterium present, it is very quickly synthesized into ^{4}He, which is so highly energetically favored that we can determine its abundance by simply counting the number of neutrons present at t_d. The helium abundance will just be $n_4 = n_n/2$, so we have $4n_4/n_N \equiv X_4 = 2n_n/n_N = 2X_n$ at $t = t_d$. To find $X_n(t_d)$ we must modulate our value of $X_n(t \to \infty)$ by the neutron decay in the time intervening t_f and t_d, so that

$$X_n(t_d) = X_n(T_f) e^{-t_d/\tau_n} \tag{10}$$

and thus the primordial helium abundance

$$X_4 = 2X_n(T_f) e^{-t_d/\tau_n} \tag{11}$$

where we have used t_d to approximate the time elapsed since freeze-out.

To predict the abundances of other light elements requires a detailed analysis of a network of two-body nuclear reactions. Esmailzadeh *et al* have shown that to good accuracy the abundances are just the fixed points of the rate equations [3]

$$\dot{Y}_k \propto \eta n_\gamma \sum_{i,j} Y_i Y_j \left\langle \sigma v \right\rangle_{ij \to kl} \tag{12}$$

which we have stated in terms of the elemental abundances $Y_i \equiv n_i/n_H$ relative to hydrogen. We will revisit the rate equations in the next section.

3 Estimating the Dependence on G_N

Now we turn to the work of Bambi *et al* on the dependency of the primordial abundances of D, ^{3}He, ^{4}He, and ^{7}Li upon the value of Newton's constant. Our first approach to varying G_N is to allow it to take on constant values at each of the stages we have discussed. During freeze out (T between 0.2 MeV and 2 MeV) G_N takes the constant value G_f, at the deuterium bottleneck (T between 0.02 MeV and 0.2 MeV) we have $G_N = G_d$, and from here forward we denote the present value of G_N as G_0. We treat the hypothetical values as small fractional variations of the present value

$$\delta G_N = \frac{G_N - G_0}{G_0}. \tag{13}$$

so that $G_N = G_0(1 + \delta G_N)$.

The expansion rate H depends explicitly on G_N as

$$\begin{aligned} H &= 1.66\sqrt{g_\star G_N}T^2 \\ &= 1.66\sqrt{g_\star G_0(1+\delta G_N)}T^2 \\ &\approx 1.66\sqrt{g_\star G_0}(1+\delta G_N/2)T^2. \end{aligned}$$

(Here we are abusing an equation derived – and verified – under the assumption that G_N is constant. In order to get away with it we must assume that G_N is varying sufficiently slowly compared to the expansion, so that the time derivatives of G_N that arise on the right hand side of (14) are negligible.) Now to find an expression for T_f we use $\Gamma_{n\leftrightarrow p}(T_f) \sim H(T_f)$ we use a carefully corrected result for $\Gamma_{n\leftrightarrow p}$ from Esmailzadeh [3] to find

$$T_f = 0.784(1 + \delta G_f)^{1/6} \text{ MeV} \tag{14}$$

We see that as G_f is greater the neutrons decouple earlier from protons and are more abundant, leading to a greater abundance of ^{4}He, and vice versa. In a recent calculation Mukhanov derives a formula for T as a function of X_D in the deuterium bottleneck regime and finds

$$T_d \approx 0.08(1 + 0.16 \log \eta_{10}) \text{ MeV} \tag{15}$$

$$t_d \approx 206(1 - 0.32 \log \eta_{10}) \text{ s} \tag{16}$$

where log is in base 10 and $\eta_{10} \equiv \eta \times 10^{10}$. Now using equation (11) with the roughest estimate $X_n(T_f)$ (equation 6) we find

$$t_d \sim 206(1 + \delta G_d)^{-1/2}(1 - 0.32 \log \eta_{10}) \text{ s}. \tag{17}$$

Now plugging our values into equation 11 and expanding to first order in our variations, we get the fractional variation $\delta Y_4 = \Delta Y_4/Y_4 = \delta X_4 = \Delta X_4/X_4$ of the helium abundance as a function of the two variations of G_0:

$$\delta Y_4 = 0.23\delta G_f + 0.09\delta G_d + 0.07 \log (\eta/\eta_{CMB}) \tag{18}$$

where η_{CMB} is the currently favored value of 6.14×10^{-10} and we have left η free to vary with our varying abundances.

To do the same for the other light elements we must consider how G_N factors into the rate equations. Bambi *et al* simply note that since $dT/dt \propto -T^3\sqrt{G_N}$, the time-rate equations (12) can be written as temperature rate equations so that

$$\frac{dY_i}{dT} \propto -\frac{\eta}{G_d^{1/2}}\frac{n_\gamma}{T^3}\frac{dY_i}{dt}. \tag{19}$$

We see that Y_2, Y_3, and Y_7 (for ^{2}H, ^{3}He, and ^{7}Li) depend on G_N through the factor of $\eta/G_d^{1/2}$. Considering the abundances as functions $Y_i(\eta, G_d)$ of η and G_d, we can rescale η and write $Y_i(\eta, G_d) = Y_i(\eta(1+\delta G_d)^{-1/2}, G_0)$. Now we can linearize this to get

$$\delta Y_i = \gamma_i(\eta_{CMB}, G_0)(\log\,(\eta/\eta_{CMB}) - 0.22\delta G_d) \tag{20}$$

where

$$\gamma_i \equiv \frac{\partial \log Y_i}{\partial \log \eta}. \tag{21}$$

For δY_7 we must add in the variation of Y_4 (neglecting the logarithmic dependence on η), since the formation of each ^{7}Li requires the consumption of a ^{4}He, so we get

$$\delta Y_7 = 0.23\delta G_f + 0.09\delta G_d + \gamma_7(\eta_{CMB}, G_0)(\log\,(\eta/\eta_{CMB}) - 0.22\delta G_d). \tag{22}$$

We have made it this far mostly on our imagination. We have found characteristics of how our four abundances must change if G_N took on slightly different constant values at the critical epochs of big bang nucleosynthesis. We have quantitatively affirmed our expectations: that the ^{4}He abundance depends on G_N mostly in the first epoch and only weakly on η, while ^{3}He and ^{2}H are sensitive to G_N directly in the second epoch and only indirectly (implicitly, via γ_i) in the first epoch. But we have also uncovered a relation between the baryon fraction and the expansion rate at the deuterium bottleneck, since we see from equation (20) that with fixed abundances ($\delta Y_i = 0$) we must have

$$\frac{\partial \log \eta}{\partial\, \delta G_d} = 1/0.22 \tag{23}$$

which gives us an observational constraint on δG_N.

We can compare our forms for δY_i to linear fits of numerical simulations of the abundances. (Bambi *et al* do not specify how the simulations were done, but there are several open source programs for simulating nucleosynthesis which might be easily enough modified in G_N.) Computing with a single constant variation of G_N throughout the nucleosynthesis stages ($\delta G_f = \delta G_d \equiv \delta G_N$), the results allowed the linear fits

$$\delta Y_4 \approx \;\; 0.35\delta G_N + 0.09\,\log\,(\eta/\eta_{CMB})$$

$$\delta Y_2 \approx \quad 0.93\delta G_N - 3.7 \log\,(\eta/\eta_{CMB})$$
$$\delta Y_3 \approx \quad 0.31\delta G_N - 1.3 \log\,(\eta/\eta_{CMB})$$
$$\delta Y_7 \approx \quad 1.4\delta G_N - 4.8 \log\,(\eta/\eta_{CMB})$$

which agree quite well with our forms, with

$$\gamma_2(\eta_{CMB}) \;= -3.7$$
$$\gamma_3(\eta_{CMB}) \;= -1.3$$
$$\gamma_7(\eta_{CMB}) \;= -4.8.$$

4 General Variation of G_N

We have seen that the abundance of each element responds differently to a variation δG_N of the Newton constant. To make a quantitative analysis of these responses we now allow G_N to be a fully fledged (small-valued, slowly-varying) function of time $G_N(T) = G_0 + \delta G_N(T)$. Then we introduce, and numerically solve for, the response functions $\rho_i(\eta, T)$ defined by the functional relationships

$$\delta Y_i\,[\eta, G_N(T)] = \int \rho_i(\eta, T)\delta G_N(T)\frac{dT}{T} \tag{24}$$

for $i = 2, 3, 4, 7$. This definition is motivated by a more general relation which characterizes the response to any small functional variation $\delta H(T)$ of the expansion rate, taken with respect to its standard value as a function of temperature. This relation is simply

$$\delta Y_i\,[\eta, H(T)] = 2\int \rho_i(\eta, T)\delta H(T)\frac{dT}{T}. \tag{25}$$

The first form is a special application of the more general one, where we are considering the variation of $H(T)$ to be caused only by the variation of G_N. The leading factor of 2 comes from our assumption that G_N is varying slowly compared to H; from our expansion of $H(T)$ in $\delta G_N(T)$ we have $\delta G_N(T) = 2\;\delta H(T)$. From the first form, which we use, we can see that $\rho_i(\eta, T)$ is the functional derivative

$$\rho_i(\eta, T) \equiv \frac{\delta\,\ln\,Y_i[\eta, \delta G_N(T)]}{\delta G_N(T)} \tag{26}$$

Bambi *et al* calculate the response functions numerically with $\eta = \eta_{CMB}$, and the results are shown in Figure 1. Each function shows two peaks corresponding in the two stages of nucleosynthesis. Within the limitations of our model, the areas under these peaks tell us about the relative dependence of each light element upon the *average* value of the Newton constant at each stage. The area under each curve gives the numerical coefficients $\delta Y_i/\delta G_N$ found at the end of the last section. We can see that our use of constants G_f and G_d makes sense if we interpret them as average values over their respective domains.

To apportion the net response according to the contribution of each δG_i, Bambi *et al* calculate the parameters

$$\alpha_i = \int_{0.2\text{ MeV}}^{2\text{ MeV}} \rho_i(\eta, T)\frac{dT}{T}$$
$$\beta_i = \int_{0.02\text{ MeV}}^{0.2\text{ MeV}} \rho_i(\eta, T)\frac{dT}{T}$$

for each element, finding

$$\alpha_4 = 0.22, \quad \beta_4 = 0.12$$
$$\alpha_2 = 0.12, \quad \beta_2 = 0.80$$
$$\alpha_3 = 0.04, \quad \beta_3 = 0.29$$
$$\alpha_7 = 0.14, \quad \beta_7 = -0.83.$$

These compare well to the coefficients of $\delta Y_i/\delta G_f$ in our predictions.

Because the abundances of ^{2}H, ^{3}He, and ^{7}Li depend almost entirely on the rate of expansion at the bottleneck epoch, the bounds on these abundances serve rather directly as bounds on δG_d. The ^{4}He abundance depends broadly on G_f and G_d; we take it to be a constraint on the quantity $0.65\delta G_f + 0.35\delta G_d$.

5 Constraining δG_N with BBN

Bambi *et al* use these bounds to determine constraints on the deviation of G_N, constant during big bang nucleosynthesis, from its present value. The results of their analysis are shown in Figure 2. The panel of interest is panel *f*, which shows the bound $\delta G_N = -0.09 \pm 0.05$. The fit has a value of $\chi^2_{min} = 1.0$, and the authors conclude that there is no observational warrant for theories in which G_N varies during the course of big bang nucleosynthesis.

References

[1] C. Bambi, M. Gianotti, and F. L. Villante. Response of primordial abundances to a general modification of g_n and/or of the early universe expansion rate. *Physical Review D*, 71(12), June 2005.

[2] J. Bernstein, L. S. Brown, and G. Feinberg. Cosmological helium production simplified. *Reviews of Modern Physics*, 61(1):25–39, January 1989.

[3] R. Esmailzadeh, G. D. Starkman, and S. Dimopoulos. Primordial nucleosynthesis without a computer. *Astrophysical Journal, Part 1*, 378:504–518, September 1991.

[4] E. W. Kolb and M. S. Turner. *The Early Universe.* Number 69 in Frontiers in Physics. Westview, 1990.

[5] V. Mukhanov. Nucleosynthesis without a computer. *eprint arXiv:astro-ph/0303073*, page 24, March 2003.

[6] S. Sarkar. Big bang nucleosynthesis and physics beyond the standard model. *Rep. Prog. Phys.*, 59:1493–1609, 1996.

[7] S. Weinberg. *Gravitation and Cosmology.* John Wiley & Sons, 1972.

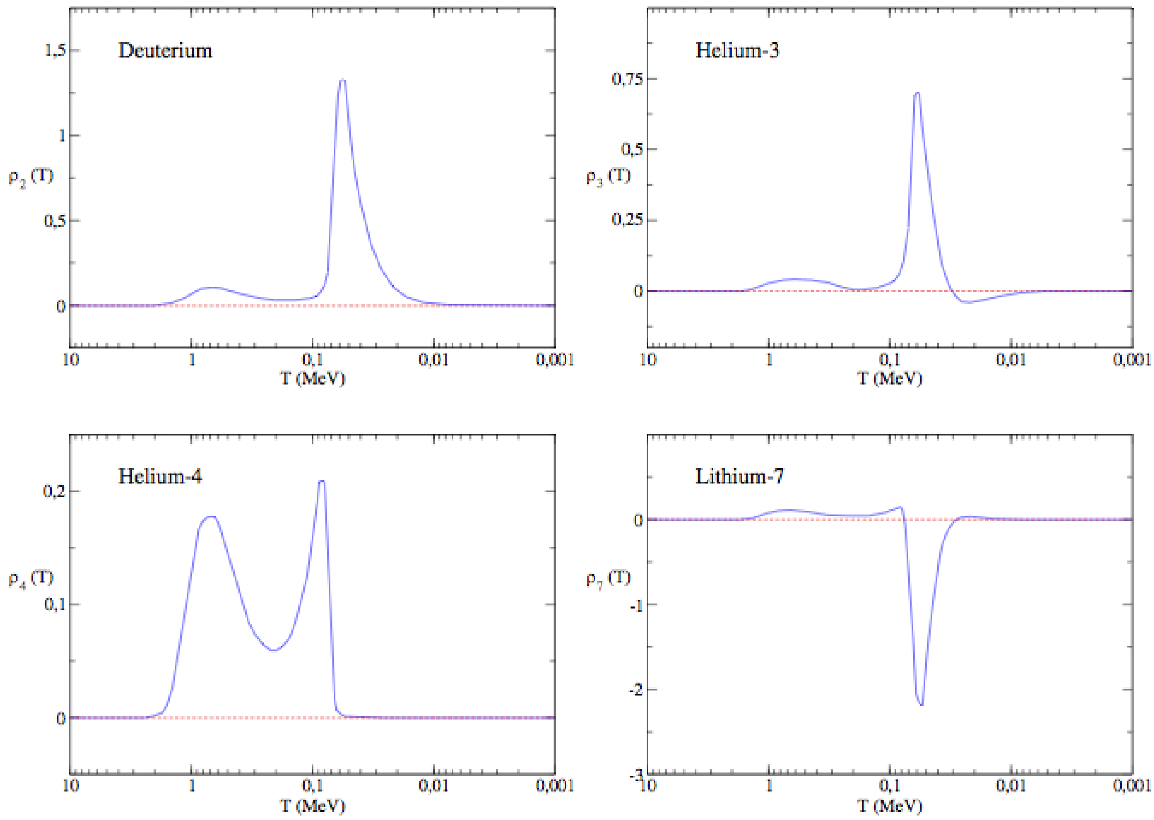

Figure 1: Response functions $\rho_i(\eta_{CMB}, T)$, versus temperature T, quantify the effects of an arbitrary time-dependent variation of the Newton constant G_N in the early universe on the formation of light elements.

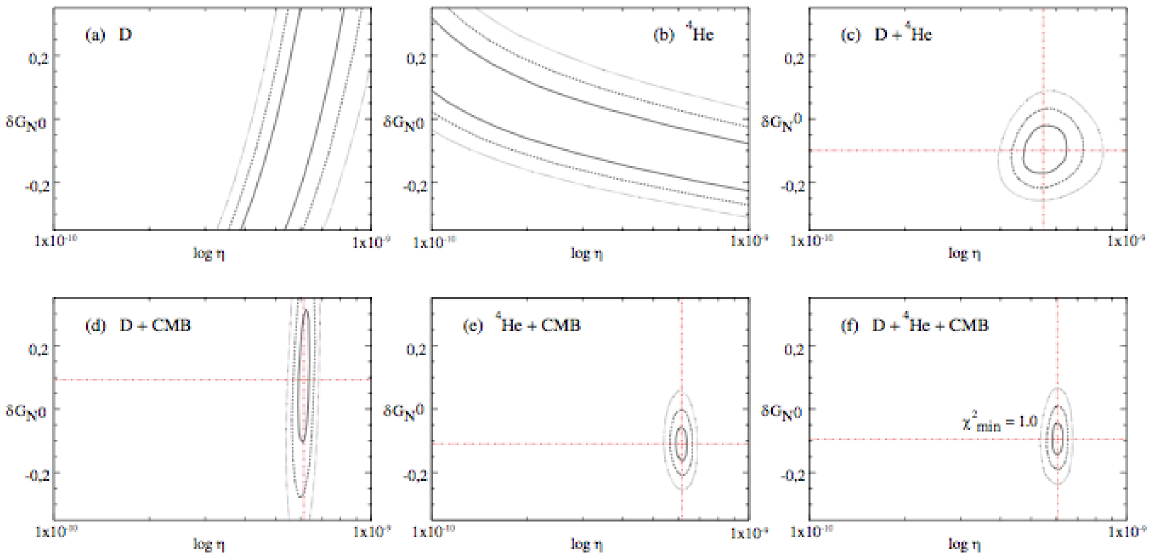

Figure 2: Bounds on δG_N and η obtained from ^{2}H and ^{4}He abundance measurements. Upper panels consider BBN constraints alone. Lower panels use constraints on η from CMB measurements.